

# Incorporating local boundary conditions into nonlocal theories <sup>\*†</sup>

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## Abstract

We study nonlocal equations from the area of peridynamics on bounded domains. In our companion paper, we discover that, on  $\mathbb{R}^n$ , the governing operator in peridynamics, which involves a convolution, is a bounded function of the classical (local) governing operator. Building on this, we define an abstract convolution operator on bounded domains which is a generalization of the standard convolution based on integrals. The abstract convolution operator is a function of the classical operator, defined by a Hilbert basis available due to the purely discrete spectrum of the latter. As governing operator of the nonlocal equation we use a function of the classical operator, this allows us to incorporate local boundary conditions into nonlocal theories. The governing operator is determined by what we call the regulating function. By choosing different regulating functions, we can define governing operators tailored to the needs of the underlying application.

For the homogeneous wave equation with the considered boundary conditions, we prove that continuity is preserved by time evolution. Namely, if the initial data is continuous, then the solution is continuous for  $t \in \mathbb{R}$ . This is due to the fact that the solution has a unique decomposition into two parts. The first part is the product of a function of time with the initial data. The second part is continuous. This decomposition is induced by the fact that the governing operator has a unique decomposition into multiple of the identity and a Hilbert-Schmidt operator. The decomposition also implies that discontinuities remain stationary.

We give explicit solution expressions for the initial value problems with prominent boundary conditions such as periodic, antiperiodic, Neumann, and Dirichlet. In order to connect to the standard convolution, we give an integral representation of the abstract convolution operator. We present additional “simple” convolutions based on periodic and antiperiodic boundary conditions that lead Neumann and Dirichlet boundary conditions.

We present a numerical study of the solutions of the wave equation. For discretization, we employ a weak formulation based on a Galerkin projection and use piecewise polynomials on each element which allows discontinuities of the approximate solution at the element borders. We study convergence order of solutions with respect to polynomial order and observe optimal convergence. We depict the solutions for each boundary condition.

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# 1 Introduction

There are indications that a development of nonlocal theories is necessary for description of certain natural phenomena. It is part of the folklore in physics that the point particle model, which is the root for *locality* in physics, is the cause of unphysical singular behavior in the description of phenomena. On the other hand, all fundamental theories of physics are local. There are many more alternatives for formulating nonlocal theories compared to local ones. Therefore, the task of formulating a viable nonlocal theory consistent with experiment seems much harder than that of a local one. In any case, for such formulation, an understanding of the capacities of nonlocal theories appears inevitable.

Considering wave phenomena only partially successfully described by a classical wave equation, it seems reasonable to expect that a more successful model can be obtained by employing the functional calculus of self-adjoint operators, i.e., by replacing the classical governing operator  $A$  by a suitable function  $f(A)$ . We call  $f$  the *regulating function*. Since classical boundary conditions (BCs) is an integral part of the classical operator, these BCs are automatically inherited by  $f(A)$ . In this way, we vision to model wave phenomena by using appropriate  $f(A)$  and, as a consequence, need to study the effect of  $f(A)$  on the solutions. One advantage of our approach is that every symmetry that commutes with  $A$  also commutes with  $f(A)$ . As a result, required invariance with respect to classical symmetries such as translation, rotation and so forth is preserved. The choice of regulating functions appropriate for the physical situation at hand is an object for future research.

We are interested in studying instances of successful modeling by nonlocal theories of phenomena that cannot be captured by local theories. There are noteworthy developments in the area of nonlocal modeling. For instance, crack propagation [59] and viscoelastic damping [10] are modeled by peridynamics (PD) and fractional derivatives, respectively, both of which are nonlocal. Similar classes of operators are used in numerous applications such as nonlocal diffusion [8, 18, 55], population models [15, 47], image processing [30, 38], particle systems [14], phase transition [7, 6], and coagulation [29]. Further applications are in the context of multiscale modeling, where PD has been shown to be an upscaling of molecular dynamics [56, 58] and has been demonstrated as a viable multiscale material model for length scales ranging from molecular dynamics to classical elasticity [9]. Also see other related engineering applications [16, 35, 37, 51, 50], the review and news articles [18, 21, 41] for a comprehensive discussion, and the recent book [43]. In addition, we witness a major effort to meet the need for a mathematical theory for PD applications and related nonlocal problems addressing, for instance, conditioning analysis, domain decomposition and variational theory [2, 3, 4], volume constraints [18], nonlinearity [23, 24, 25, 42], discretization [1, 4, 28, 63], numerical methods [17, 19, 22, 54], and various other aspects [5, 20, 26, 27, 33, 39, 40, 44, 45, 46, 56, 57, 65].

Disturbances in solids propagate in form of waves. The wave equation is the basic model for the description of the evolution of deformations. This paper focuses on the class of nonlocal wave equations from PD. Classical elasticity has been successful in characterizing and measuring the resistance of materials to crack growth. On the other hand, PD, a nonlocal extension of continuum mechanics developed by Silling [59], is capable of quantitatively predicting the dynamics of propagating cracks, including bifurcation. Its effectiveness has been established in sophisticated applications such as successful description of results of Kalthoff-Winkler experiments of the fracture of a steel plate with notches [34, 60], fracture and failure of composites, nanofiber networks, and polycrystal fracture [36, 49, 62, 61]. Since PD is a nonlocal theory, one might expect only the appearance of nonlocal BCs, and, indeed, so far the concept of local BC does not apply to PD. Instead, external forces must be supplied through the loading force density  $b$  [59]. On the other hand, we demonstrate that the anticipation that local BCs are incompatible with nonlocal operators is not quite correct.

The rest of the article is structured as follows. In Section 2.1, we define convolutions on Hilbert spaces and study properties of related operators. In Section 2.2, we provide diagonalizations of the classical governing operator  $A$  and functions,  $f(A)$ , of  $A$ . The governing nonlocal operator is a function of  $A$ . On the other hand, the representation of the solution of the initial value problem corresponding to  $f(A)$  contains bounded functions of  $f(A)$ . Hence, in Section 2.3, we prove that bounded functions of  $f(A)$  are bounded functions of  $A$ . In Section 2.4, we give a representation of the solution of the inhomogeneous wave equation in terms of an eigenbasis. In addition, in Section 2.5, we make the connection to functions of  $A$  from the peridynamic governing operator in the unbounded domain case.

In Section 3, we address the important question under what conditions solutions will satisfy prescribed BCs. We find that Hilbert-Schmidt operators play a crucial role in satisfying BCs. The

Hilbert-Schmidt property leads to a uniform convergence argument which allows us to interchange limits. Hence, BCs are automatically satisfied. In addition, the Hilbert-Schmidt property leads to *smoothing* of the input in the sense an  $L^2$  function is mapped into a function that is continuous up to the boundary. With additional decay conditions of the eigenvalues, we reach a uniform convergence argument also for derivatives which allows us to interchange limits. As a consequence, BCs that involve derivatives are also automatically satisfied.

In Section 4, we apply the material from previous sections to study prominent BCs such as periodic, antiperiodic, Neumann, and Dirichlet BCs. In the case of periodic and antiperiodic BCs, integral representations of the abstract convolutions are relatively straightforward to establish. On the other hand, for Neumann boundary condition, this representation is considerably more involved, requiring arguments related to half-way symmetry of functions. For Dirichlet BC, we give representation in terms limits of integral convolutions. For both Neumann and Dirichlet conditions, we also give other plausible simple definitions of convolutions that are related to periodic and antiperiodic extensions of the micromodulus function.

In Section 5, we present a comprehensive numerical treatment of the nonlocal wave equation. We have two goals in numerical experiments. First, we want to demonstrate that discontinuities of the initial data remain stationary for  $t \in \mathbb{R}$ . Second, solutions satisfy the BCs also for  $t \in \mathbb{R}$ . In order to show that the two goals are accomplished, we choose discontinuous initial data and run experiments showing wave evolutions for all of the considered BCs; periodic, antiperiodic, Neumann, and Dirichlet. Furthermore, by choosing continuous initial data, we draw parallels between the local and nonlocal wave equations for Neumann and Dirichlet BCs.

## 2 Construction for the Bounded Domain

Practical applications call for a bounded domain. In the unbounded domain case, in the companion paper [12], we discovered that PD uses as governing operator a function  $f(A)$  of the classical operator  $A$ :

$$\begin{aligned} f(A)u(x, t) &:= \int_{\mathbb{R}^n} C(y - x) \cdot (u(y, t) - u(x, t)) dy \\ &= \left( \int_{\mathbb{R}^n} C(y) dy \right) u(x, t) - \int_{\mathbb{R}^n} C(x - y) u(y, t) dy. \end{aligned} \quad (2.1)$$

The question is the generalization of convolutions to functions on bounded domains, preserving the Banach algebra structure of convolutions for functions on  $L^1(\mathbb{R}^n)$ . Indeed, such a generalization is known for periodic functions. In Section 4.1, we show that this definition is a special case of Theorem 1.

### 2.1 Convolutions on Hilbert Spaces

In the unbounded domain case, in our companion paper [12], we discovered that the governing nonlocal operator is a function of a multiple of the Laplace operator, the classical governing operator. Therefore, for the bounded domain case, it is natural to define the governing operator as a function of the corresponding classical (local) operator. This opens a gateway to incorporate local BCs to nonlocal theories.

For simplicity, we choose the classical (local) operator to be a multiple of the Laplace operator with appropriate BCs. In the bounded domain case, the spectrum of the Laplace operator with classical BCs such as periodic, antiperiodic, Neumann, Dirichlet, and Robin is purely discrete. Furthermore, we can explicitly calculate the eigenfunctions  $e_k$  corresponding to each BC and the subscript signifies the BC used;  $\text{BC} \in \{\text{p}, \text{a}, \text{N}, \text{D}\}$  where p, a, N, and D stand for periodic, antiperiodic, Neumann, Dirichlet, respectively. These eigenfunctions form a Hilbert basis (complete orthonormal basis) through which the abstract convolution can be defined as follows:

$$C *_{\text{BC}} u := \sum \langle e_k | C \rangle \langle e_k | u \rangle e_k, \quad (2.2)$$

where

$$\langle e_k | u \rangle := \int_{-1}^1 e_k^*(y) u(y) dy.$$

The nonlocal wave equation we solve is given as follows:

$$u_{tt}(x, t) + \varphi(A_{\text{BC}})u(x, t) = 0, \quad x \in (-1, 1), \quad t \in [0, T], \quad (2.3)$$

where  $T \geq 0$ ,  $\varphi : \sigma(A_{\text{BC}}) \rightarrow \mathbb{R}$  is a bounded function and  $A_{\text{BC}}$  is the (local) classical operator with spectrum  $\sigma(A_{\text{BC}})$ . For instance, the convolution in (2.2) defines the governing operator  $c - C *_{\text{BC}}$  where  $c$  is an appropriate constant. The *regulating function*  $\varphi : \sigma(A_{\text{BC}}) \rightarrow \mathbb{R}$  is determined by the following equation:

$$\varphi(A_{\text{BC}}) := c - C *_{\text{BC}}. \quad (2.4)$$

Representing  $u$  in this Hilbert basis,

$$u = \sum \langle e_k | u \rangle e_k,$$

we arrive at

$$(c - C *_{\text{BC}})u = \sum [c - \langle e_k | C \rangle] \langle e_k | u \rangle e_k = \sum \varphi(\lambda_k) \langle e_k | u \rangle e_k,$$

where  $(\lambda_k, e_k)$  denotes an eigenpair of the classical operator. Since the last relation regulates which function of the classical operator is used to define the nonlocal operator, for simplicity, we also call  $\varphi_C(k) := \varphi(\lambda_k)$  the *regulating function*:

$$\varphi_C(k) = c - \langle e_k | C \rangle. \quad (2.5)$$

The following is a formal definition of convolutions on Hilbert spaces which leads to a Banach algebra structure.

**Theorem 1. (Convolutions on Hilbert Spaces)** Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $(X, \langle | \rangle)$  a non-trivial  $\mathbb{K}$ -Hilbert space with induced norm  $\| \cdot \|$  and  $M \subset X$  a Hilbert basis. By

$$\xi * \eta := \sum_{e \in M(\xi) \cap M(\eta)} \langle e | \xi \rangle \langle e | \eta \rangle \cdot e$$

for  $\xi, \eta \in X$ , where

$$M(\xi) := \{e \in M : \langle e | \xi \rangle \neq 0\}, \quad M(\eta) := \{e \in M : \langle e | \eta \rangle \neq 0\},$$

and the sum over the empty set is defined as  $0_X$ , there is defined a bilinear, commutative and associative map

$$\left( \begin{array}{l} * : X^2 \rightarrow X \\ (\xi, \eta) \mapsto \xi * \eta \end{array} \right)$$

such that

$$\|\xi * \eta\| \leq \|\xi\| \cdot \|\eta\|$$

for all  $\xi, \eta \in X$  and, as a consequence,  $(X, +, \cdot, *, \| \cdot \|)$  is a commutative Banach algebra.

*Proof.* We note for  $\xi, \eta \in X$  that, as intersection of at most countable subsets of  $X$ ,  $M(\xi) \cap M(\eta)$  is at most countable. Furthermore, for every finite subset  $S$  of  $M(\xi) \cap M(\eta)$

$$\begin{aligned} \sum_{e \in S} |\langle e | \xi \rangle \langle e | \eta \rangle|^2 &= \sum_{e \in S} |\langle e | \xi \rangle|^2 \cdot |\langle e | \eta \rangle|^2 \leq \left( \sum_{e \in S} |\langle e | \xi \rangle|^2 \right) \cdot \left( \sum_{e \in S} |\langle e | \eta \rangle|^2 \right) \\ &\leq \left( \sum_{e \in M(\xi)} |\langle e | \xi \rangle|^2 \right) \cdot \left( \sum_{e \in M(\eta)} |\langle e | \eta \rangle|^2 \right) = \|\xi\|^2 \cdot \|\eta\|^2. \end{aligned}$$

Hence the sequence

$$(|\langle e | \xi \rangle \langle e | \eta \rangle|^2)_{e \in M(\xi) \cap M(\eta)}$$

is (absolutely) summable. As a consequence, the sequence

$$(\langle e | \xi \rangle \langle e | \eta \rangle \cdot e)_{e \in M(\xi) \cap M(\eta)}$$

is summable, with a sum that is independent of the order of summation. Therefore,  $\xi * \eta$  is well-defined and satisfies

$$\|\xi * \eta\|^2 = \sum_{e \in M(\xi) \cap M(\eta)} |\langle e | \xi \rangle \langle e | \eta \rangle|^2 \leq \|\xi\|^2 \cdot \|\eta\|^2.$$

That  $*$  is bilinear, commutative and associative is obvious.  $\square$

We present properties of operators induced by convolutions on Hilbert spaces.

**Corollary 2.** *Let  $\mathbb{K}$ ,  $(X, \langle | \rangle)$ ,  $\| \cdot \|$ ,  $M$  and  $*$  as in Theorem 1. Then for every  $\xi \in X$ ,  $\xi * \cdot \in L(X, X)$  and, in addition, Hilbert-Schmidt and therefore also compact.*

*Proof.* Let  $\xi \in X$ . First, it follows from Theorem 1 that  $\xi * \cdot \in L(X, X)$ . Furthermore, from the definition of  $*$ , it follows for every  $e \in M$  that

$$\xi * e = \langle e | \xi \rangle . e .$$

In particular, this implies that the set

$$M(A) := \{e \in M : \xi * e \neq 0\}$$

is at most countable and that

$$(\| \langle e | \xi \rangle . e \|^2)_{e \in M(A)} = (| \langle e | \xi \rangle |^2)_{e \in M(A)}$$

is summable. Hence  $\xi * \cdot$  is in addition Hilbert-Schmidt and therefore also compact.  $\square$

**Remark 3.** *In the context of  $L^2$  spaces, the Hilbert-Schmidt property leads to smoothing of input functions; see Theorem 23.*

In the case that the Hilbert basis is an eigenbasis of an operator  $A$ , we give a condition that the operators in Corollary 2 are functions of  $A$ .

**Corollary 4.** *Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $(X, \langle | \rangle)$  a non-trivial  $\mathbb{K}$ -Hilbert space, with induced norm  $\| \cdot \|$ ,  $A$  a densely-defined, linear and self-adjoint operator in  $X$  with a purely discrete spectrum  $\sigma(A)$ ,  $M \subset X$  the Hilbert basis consisting of the eigenvectors of  $A$  and, finally,  $*$  the convolution corresponding to  $M$ . Then, for every  $\xi \in X$ , satisfying*

$$\langle e | \xi \rangle = \langle e' | \xi \rangle$$

*for every  $e, e' \in M$  corresponding to the same eigenvalue of  $A$ ,  $\xi * \cdot$  is a bounded function of  $A$ .*

*Proof.* Let  $\xi \in X$  and for every  $e \in M$ ,  $\lambda(e) \in \mathbb{R}$  the corresponding eigenvalue of  $A$ . Then the spectrum  $\sigma(A)$  of  $A$  is given by

$$\sigma(A) := \{\lambda(e) : e \in M\} .$$

We define  $f \in B(\sigma(A), \mathbb{C})$ , where  $B(\sigma(A), \mathbb{C})$  denotes complex valued bounded functions on  $\sigma(A)$ , by

$$f(\lambda(e)) := \langle e | \xi \rangle$$

for every  $e \in M$ . This definition leads to a well-defined  $f$ , since according to the assumptions,

$$\langle e | \xi \rangle = \langle e' | \xi \rangle$$

for every  $e, e' \in M$  satisfying  $\lambda(e) = \lambda(e')$ . Also, we note that  $f$  is bounded since

$$|f(\lambda(e))| = | \langle e | \xi \rangle | \leq \| \xi \| ,$$

for every  $e \in M$ . Furthermore, from the spectral theorem for densely-defined, linear and self-adjoint Hilbert spaces and the definition of  $*$ , it follows for every  $e \in M$  that

$$f(A)e = f(\lambda(e))e = \langle e | \xi \rangle . e = \xi * e$$

and hence that

$$\xi * \cdot = f(A) .$$

$\square$

We note that simple rotations of the eigenbasis lead to different convolutions.

**Remark 5.** Note that if  $M \subset X$  is a Hilbert basis and  $\alpha : M \rightarrow S^1$ , where  $S^1 \subset \mathbb{C}$  denotes the unit circle, then

$$M_\alpha := \{\alpha(e) \cdot e : e \in M\}$$

is also a Hilbert basis, and for every  $\xi, \eta \in X$ ,

$$\xi *_\alpha \eta = \sum_{e \in M(\xi) \cap M(\eta)} (\alpha(e))^* \langle e | \xi \rangle \langle e | \eta \rangle \cdot e ,$$

where  $*_\alpha$  denotes the convolution that is associated to  $M_\alpha$ .

To avoid repetition in the upcoming discussions, we make the following assumptions with corresponding abstract homogeneous governing equation:

$$u''(t) + f(A)u(t) = 0. \quad (2.6)$$

On the other hand, in motivations we use symbols from the governing equation (2.3) such as  $\varphi$  and  $A_{\text{BC}}$  instead of  $f$  and  $A$ .

**Assumption 6.** In the following, let  $(X, \langle | \rangle)$  be a non-trivial complex Hilbert Space,  $A : D(A) \rightarrow X$  a densely-defined, linear and self-adjoint operator with a purely discrete spectrum  $\sigma(A)$ , i.e., such that the (non-empty, closed and real)  $\sigma(A)$  is discrete and contains only eigenvalues of finite multiplicity. Note that this implies that,  $\sigma(A)$  is at most countable, there is an at most countable Hilbert basis  $M \subset X$  of  $X$  consisting of eigenvectors of  $A$  and that  $(X, \langle | \rangle)$  is separable. In the following, we consider the particular case that  $X$  is infinite dimensional and hence that  $M$  is countable. As a consequence,

$$M = \{e_1, e_2, \dots\} ,$$

where  $e_1, e_2, \dots$  are pairwise orthogonal normalized eigenvectors of  $A$ . In particular, for every  $k \in \mathbb{N}^*$ , let  $\lambda_k$  be the eigenvalue corresponding to  $e_k$ . Then

$$\sigma(A) = \{\lambda_k : k \in \mathbb{N}^*\} .$$

## 2.2 Diagonalization of $A$ and Induced Representation of Bounded Functions of $A$

With the knowledge of the eigenbasis, the diagonalization of  $A$  is straightforward.

**Theorem 7.** Define  $U : L_{\mathbb{C}}^2(I) \rightarrow \sum_{k \in \mathbb{N}^*} \mathbb{C} = l_{\mathbb{C}}^2$  by

$$U(\xi) := (\langle e_k | \xi \rangle_2)_{k \in \mathbb{N}^*}$$

for every  $\xi \in X$ . Then  $U$  is a Hilbert space isomorphism. In particular,

$$U \circ A \circ U^{-1} = \sum_{k \in \mathbb{N}^*} \lambda_k \cdot \text{id}_{\mathbb{C}} . \quad (2.7)$$

*Proof.* In particular, for every  $\xi \in D(A)$

$$\begin{aligned} U \circ A \circ U^{-1}(\langle e_k | \xi \rangle_2)_{k \in \mathbb{N}^*} &= U \circ A \xi = (\lambda_k \langle e_k | \xi \rangle_2)_{k \in \mathbb{N}^*} \\ &= \left[ \sum_{k \in \mathbb{N}^*} \lambda_k \cdot \text{id}_{\mathbb{C}} \right] (\langle e_k | \xi \rangle_2)_{k \in \mathbb{N}^*} . \end{aligned} \quad (2.8)$$

For the proof of the latter, we note for every  $k \in \mathbb{N}^*$  that

$$\langle e_k | A \xi \rangle_2 = \langle A e_k | \xi \rangle_2 = \lambda_k \langle e_k | \xi \rangle_2 .$$

The identity (2.8) implies that

$$\sum_{k \in \mathbb{N}^*} \lambda_k \cdot \text{id}_{\mathbb{C}} \supset U \circ A \circ U^{-1}$$

and hence, since  $U \circ A \circ U^{-1}$  and  $\sum_{k \in \mathbb{N}^*} (\lambda_k \cdot \text{id}_{\mathbb{C}})$  are both densely-defined, linear and self-adjoint operators in  $l_{\mathbb{C}}^2$ , that

$$U \circ A \circ U^{-1} = \sum_{k \in \mathbb{N}^*} \lambda_k \cdot \text{id}_{\mathbb{C}} .$$

□

The Hilbert space isomorphism  $U$  in Theorem 7 also diagonalizes bounded functions of  $A$ . In the solution of the initial value problem of the wave equation with governing operator  $A$ , only bounded functions appear. Namely, for  $f \in B(\sigma(A), \mathbb{C})$  and  $\xi \in X$ ,

$$f(A) \xi = \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | \xi \rangle_2 \cdot e_k. \quad (2.9)$$

### 2.3 Functions of Functions of $A$

The governing operator in the nonlocal equation (2.3) is a bounded function,  $\varphi(A_{\text{BC}})$ , of the classical operator  $A_{\text{BC}}$ . As a consequence, in the solution of the initial value problem, bounded functions,  $g(\varphi(A_{\text{BC}}))$ , of bounded function,  $\varphi(A_{\text{BC}})$ , appear. Since  $A_{\text{BC}}$  has a purely discrete spectrum, it is easy to see that  $g(\varphi(A_{\text{BC}}))$  is a bounded function of  $A_{\text{BC}}$ ,  $(g \circ \varphi)(A_{\text{BC}})$ , as indicated in the following.

**Theorem 8.** *Let  $f \in B(\sigma(A), \mathbb{R})$ .*

(i) *Then  $f(A)$  is self-adjoint with pure point spectrum,  $\sigma(f(A))$ , given by*

$$\sigma(f(A)) = \overline{\{f(\lambda_k) : k \in \mathbb{N}^*\}}.$$

(i) *If  $g \in B(\sigma(f(A)), \mathbb{C})$ ,  $k \in \mathbb{N}^*$  and  $\eta \in X$ , then*

$$g(f(A)) = (g \circ f)(A). \quad (2.10)$$

*Proof.* If  $f \in B(\sigma(A), \mathbb{R})$ , we conclude from the spectral theorem for densely-defined, linear and self-adjoint Hilbert spaces, that  $f(A)$  is, in particular, self-adjoint and from (2.9) that every member of the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$  is an eigenvector of  $f(A)$ . Hence  $f(A)$  has a *pure point spectrum*, and its spectrum  $\sigma(f(A))$  is given by

$$\sigma(f(A)) = \overline{\{f(\lambda_k) : k \in \mathbb{N}^*\}}.$$

Furthermore, if  $g \in B(\sigma(f(A)), \mathbb{C})$ ,  $k \in \mathbb{N}^*$  and  $\eta \in X$ , it follows from the spectral theorem for densely-defined, linear and self-adjoint Hilbert spaces that

$$\begin{aligned} g(f(A)) e_k &= g(f(\lambda_k)) \cdot e_k, \\ g(f(A)) \eta &= g(f(A)) \sum_{k=1}^{\infty} \langle e_k | \eta \rangle_2 \cdot e_k = \sum_{k=1}^{\infty} \langle e_k | \eta \rangle_2 \cdot g(f(A)) e_k \\ &= \sum_{k=1}^{\infty} g(f(\lambda_k)) \langle e_k | \eta \rangle_2 \cdot e_k \\ &= \sum_{k=1}^{\infty} (g \circ f)(\lambda_k) \langle e_k | \eta \rangle_2 \cdot e_k = (g \circ f)(A) \eta \end{aligned}$$

and hence that

$$g(f(A)) = (g \circ f)(A). \quad (2.11)$$

□

The following functions appear in the solution of the initial value problem for (2.6) with abstract governing operator  $f(A)$ .

**Remark 9.** *In particular, for all  $t \in \mathbb{R}$ ,  $\eta \in X$ ,*

$$\begin{aligned} \left[ \overline{\cos(t\sqrt{\cdot})} \Big|_{\sigma(f(A))} \right] (f(A)) \eta &= \sum_{k=1}^{\infty} \overline{\cos(t\sqrt{\cdot})} (f(\lambda_k)) \langle e_k | \eta \rangle_2 \cdot e_k, \\ \left[ \frac{\overline{\sin(t\sqrt{\cdot})}}{\sqrt{\cdot}} \Big|_{\sigma(f(A))} \right] (f(A)) \eta &= \sum_{k=1}^{\infty} \frac{\overline{\sin(t\sqrt{\cdot})}}{\sqrt{\cdot}} (f(\lambda_k)) \langle e_k | \eta \rangle_2 \cdot e_k. \end{aligned}$$

## 2.4 Solution of the Inhomogeneous Wave Equation

A solution for vanishing initial data, of the inhomogeneous wave equation with abstract governing operator  $f(A)$  corresponding to continuous inhomogeneity is given by the following theorem.

**Theorem 10.** *In addition, let  $A : D(A) \rightarrow X$  be positive and*

$$b : \mathbb{R} \rightarrow L_{\mathbb{C}}^2(I)$$

*be continuous. Furthermore, let  $f : \sigma(A) \rightarrow \mathbb{R}$  be bounded. Then,  $v : \mathbb{R} \rightarrow X$ , for every defined by*

$$v(t) := \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} \Big|_{\sigma(f(A))} \right] (f(A)) b(\tau) d\tau ,$$

*where  $\int$  denotes weak integration in  $X$ ,*

$$I_t := \begin{cases} [0, t] & \text{if } t \geq 0 \\ [t, 0] & \text{if } t < 0 \end{cases} ,$$

*is twice continuously differentiable, such that*

$$v(0) = v'(0) = 0 ,$$

*and*

$$v''(t) + A v(t) = b(t), \quad t \in \mathbb{R}.$$

*Proof.* See the proof in the companion paper [12, Thm. 2.5]. □

The general inhomogeneous solution is given by the superposition of the general homogeneous solution with  $v$  in (2.12).

**Remark 11.** *For every  $k \in \mathbb{N}^*$ , we calculate the expansion coefficients  $\langle e_k | v(t) \rangle_2$  with respect to Hilbert basis corresponding to  $A$ :*

$$\langle e_k | v(t) \rangle_2 = \int_0^t \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (f(\lambda_k)) \langle e_k | b(\tau) \rangle d\tau.$$

*Hence,*

$$v(t) = \sum_{k=1}^{\infty} \left\{ \int_0^t \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (f(\lambda_k)) \langle e_k | b(\tau) \rangle d\tau \right\} e_k . \quad (2.12)$$

## 2.5 The Case of Nonlocal Governing Operators Involving Convolutions

Let  $*$  denote the convolution in  $X$  that, according to Theorem 1, is associated to the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$ . We consider operators that are analogous to peridynamic governing operators in the unbounded domain case and make the connection to functions of  $A$ .

**Theorem 12.** *Let  $C \in X$  and  $c \in \mathbb{R}$ . Then, the following holds.*

(i)  $c - C * \cdot$  is a bounded operator.

(ii)  $c - C * \cdot$  is self-adjoint if and only if

$$\langle e_k | C \rangle$$

*is real for every  $k \in \mathbb{N}^*$  and, if self-adjoint, positive, if and only if*

$$\langle e_k | C \rangle \leq c$$

*for every  $k \in \mathbb{N}^*$ .*



(iii) If in addition,

$$\langle e_k | C \rangle = \langle e_l | C \rangle$$

for every  $k, l \in \mathbb{N}^*$  such that  $\lambda_k = \lambda_l$ , then

$$c - C * \cdot = (c - f)(A) ,$$

where  $f \in B(\sigma(A), \mathbb{C})$  is defined by

$$f(\lambda_k) := \langle e_k | C \rangle .$$

*Proof.* For  $C, \xi, \eta \in X$  and  $c \in \mathbb{R}$ ,

$$c - C * \cdot$$

defines a linear operator in  $X$  that, as a consequence of

$$\|(c - C*)\xi\| = \|c.\xi - C * \xi\| \leq |c| \cdot \|\xi\| + \|C * \xi\| \leq (|c| + \|C\|) \cdot \|\xi\| ,$$

is bounded. Furthermore,

$$\begin{aligned} C * \xi &= \sum_{k \in \mathbb{N}^*} \langle e_k | C \rangle_2 \langle e_k | \xi \rangle \cdot e_k , \quad c.\xi = \sum_{k \in \mathbb{N}^*} c \langle e_k | \xi \rangle \cdot e_k , \\ c.\xi - C * \xi &= \sum_{k \in \mathbb{N}^*} (c - \langle e_k | C \rangle) \langle e_k | \xi \rangle \cdot e_k , \\ U(c.\xi - C * \xi) &= \left( (c - \langle e_k | C \rangle) \langle e_k | \xi \rangle \right)_{k \in \mathbb{N}^*} , \\ U(c - C * \cdot)U^{-1} &= \sum_{k \in \mathbb{N}^*} (c - \langle e_k | C \rangle) \cdot \text{id}_{\mathbb{C}} . \end{aligned}$$

Therefore,  $c - C * \cdot$  is self-adjoint if and only if

$$\langle e_k | C \rangle$$

is real for every  $k \in \mathbb{N}^*$  and, is self-adjoint, positive, if and only if

$$\langle e_k | C \rangle \leq c$$

for every  $k \in \mathbb{N}^*$ . If in addition,

$$\langle e_k | C \rangle = \langle e_l | C \rangle$$

for every  $k, l \in \mathbb{N}^*$  such that  $\lambda_k = \lambda_l$ , we conclude from the proof of Corollary 4 that

$$c - C * \cdot = (c - f)(A) ,$$

where  $f \in B(\sigma(A), \mathbb{C})$  is defined by

$$f(\lambda_k) := \langle e_k | C \rangle$$

for every  $k \in \mathbb{N}^*$ . □

**Remark 13.** If  $c - C * \cdot$  is self-adjoint and positive, then, for all  $t \in \mathbb{R}$ ,  $\eta \in X$ , it follows from (2.11) that

$$\begin{aligned} \left[ \overline{\cos(t\sqrt{\cdot})} \Big|_{\sigma(c - C*)} \right] (c - C * \cdot) \eta &= \sum_{k=1}^{\infty} \overline{\cos(t\sqrt{\cdot})} (c - \langle e_k | C \rangle) \langle e_k | \eta \rangle_2 \cdot e_k , \\ \left[ \frac{\overline{\sin(t\sqrt{\cdot})}}{\sqrt{\cdot}} \Big|_{\sigma(c - C*)} \right] (c - C * \cdot) \eta &= \sum_{k=1}^{\infty} \frac{\overline{\sin(t\sqrt{\cdot})}}{\sqrt{\cdot}} (c - \langle e_k | C \rangle) \langle e_k | \eta \rangle_2 \cdot e_k . \end{aligned}$$

Finally, the expression of  $v$  follows from (2.12):

$$v(t) = \sum_{k=1}^{\infty} \left\{ \int_0^t \frac{\overline{\sin((t-\tau)\sqrt{\cdot})}}{\sqrt{\cdot}} (c - \langle e_k | C \rangle) \langle e_k | b(\tau) \rangle d\tau \right\} e_k .$$

### 3 Smoothing Functions of Operators and Boundary Conditions

For motivation, we consider the Dirichlet eigenfunction expansion of  $u = \chi_{[-1/2, 1/2]} + 1$  on the interval  $I = [-1, 1]$

$$u = \sum_{k=1}^{\infty} \langle e_k^{\mathbb{D}} | u \rangle_2 e_k^{\mathbb{D}}.$$

Although

$$\sum_{k=1}^N \langle e_k^{\mathbb{D}} | u \rangle_2 e_k^{\mathbb{D}}$$

is infinitely differentiable on  $I$  and satisfies the Dirichlet BCs, we find that  $u$  is neither continuous nor satisfies the BCs. This opens the important question under what conditions the solution will satisfy the BCs. We address this question in this section and find that Hilbert-Schmidt operators play a crucial role in satisfying the BCs. The basis for this is provided by the fact that the governing operators in Section 4 are of the form  $c - C$  where  $c \in \mathbb{R}$  and  $C$  is Hilbert-Schmidt operator. Consequently, the assumptions on the Hilbert-Schmidt property made in the following apply to all cases discussed in Section 4. BCs involving derivatives require stronger conditions on the decay of the eigenvalues of the operator  $C$  than that of provided by the Hilbert-Schmidt property. Indeed, we find that this strong decay is satisfied in the case of Neumann BCs in Section 4.3.

#### 3.1 Strategy to Satisfy the Boundary Conditions

The solution is explicitly given in terms of the governing operator  $c - C$  as follows [12, Thm. 2.1]:

$$u(x, t) = \left[ \overline{\cos(t\sqrt{\phantom{x}})} \Big|_{\sigma(c-C)} \right] (c - C)u(x, 0) + \left[ \frac{\overline{\sin(t\sqrt{\phantom{x}})}}{\sqrt{\phantom{x}}} \Big|_{\sigma(c-C)} \right] (c - C)u_t(x, 0) \quad (3.1)$$

for all  $t \in \mathbb{R}$ , where

$$\overline{\cos(t\sqrt{\phantom{x}})} \quad \text{and} \quad \frac{\overline{\sin(t\sqrt{\phantom{x}})}}{\sqrt{\phantom{x}}}$$

denote the unique extensions of  $\cos(t\sqrt{\phantom{x}})$  and  $\sin(t\sqrt{\phantom{x}})/\sqrt{\phantom{x}}$ , respectively, to entire holomorphic functions. These functions are called *solution operators*. For brevity of discussion, let us denote either one of these solution operators as  $g(c - C)$ . We follow a two-step strategy to show how BCs are going to be satisfied:

1. Decompose the solution operator as follows:

$$g(c - C) = [g(c - C) - g(c)] + g(c), \quad (3.2)$$

so that  $g(c - C) - g(c)$  becomes a Hilbert-Schmidt operator because  $C$  is Hilbert-Schmidt. This leads to a uniform convergence argument which allows us to interchange limits; see Theorem 23 and Corollary 24. We immediately see that  $g(c - C) - g(c)$  part enforces the BCs.

2. For the remaining part  $g(c)$ , we choose initial data  $u(x, 0)$  and  $u_t(x, 0)$  that satisfy the BCs.

In order to show that  $g(c - C) - g(c)$  is Hilbert-Schmidt when  $C$  is Hilbert-Schmidt, we can utilize power series expansions. The operator  $c$  commutes with any operator  $Z$ . Let us define  $h(Z) := g(c - Z)$ . Since  $g(Z)$  is entire, so is  $h(Z)$ . Furthermore, we have a power series representation of  $g(c - Z)$  in powers of  $Z$  as follows:

$$g(c - Z) = h(Z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} Z^k,$$

where  $h^{(k)}(Z) = (-1)^k g^{(k)}(c - Z)$ . Hence,

$$g(c - C) = \sum_{k=0}^{\infty} \frac{(-1)^k g^{(k)}(c)}{k!} C^k.$$

Consequently, we have an expression for  $g(c - C) - g(c)$  that contains strictly positive powers of  $C$ :

$$g(c - C) - g(c) = \sum_{k=1}^{\infty} \frac{(-1)^k g^{(k)}(c)}{k!} C^k.$$

We have shown in Corollary 2 that abstract convolution operators are Hilbert-Schmidt. Since  $C$  is Hilbert-Schmidt due to the definition by abstract convolution, any power series in  $C$  that contains strictly positive powers is also Hilbert-Schmidt. Consequently,  $g(c - C) - g(c)$  is Hilbert-Schmidt. For details, see Lemma 16.

**Remark 14.** *Since the governing operator  $\varphi(A_{\text{BC}}) = c - C$  is a perturbation of the multiple of the identity operator by a compact operator, the static form of the inhomogeneous governing equation (2.3) satisfies the Fredholm alternative.*

### 3.2 Tools to Establish Hilbert-Schmidt Property

We start with reminding the reader of the holomorphic functional calculus from the companion paper [12] and the fact that the functions,  $\overline{\cos(t\sqrt{\cdot})}$  and  $\frac{\sin(t\sqrt{\cdot})}{\sqrt{\cdot}}$ , appearing in the solution of the initial value problem in (3.1) are entire functions.

**Lemma 15. (Holomorphic Functional Calculus)** *Let  $(X, \langle \cdot | \cdot \rangle)$  be a non-trivial complex Hilbert space,  $A \in L(X, X)$  self-adjoint and  $\sigma(A) \subset \mathbb{R}$  the (non-empty, compact) spectrum of  $A$ . Furthermore, let  $R > \|A\|$  and  $g : U_R(0) \rightarrow \mathbb{C}$  be holomorphic. Then, the sequence*

$$\left( \frac{g^{(k)}(0)}{k!} \cdot A^k \right)_{k \in \mathbb{N}}$$

*is absolutely summable in  $L(X, X)$  and*

$$(g|_{\sigma(A)})(A) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \cdot A^k.$$

*Proof.* See [12]. □

In addition, if  $A$  is Hilbert-Schmidt, more can be said. Namely, if  $g(0) = 0$ , then  $g(A)$  is Hilbert-Schmidt.

**Lemma 16. (Holomorphic Functional Calculus for Hilbert-Schmidt Operators)** *Let  $(X, \langle \cdot | \cdot \rangle)$  be a non-trivial complex Hilbert space and  $\mathcal{I}_2$  be the complex Hilbert space consisting of the Hilbert-Schmidt operators on  $X$  with induced norm  $\|\cdot\|_2$ . Furthermore, let  $A \in \mathcal{I}_2$  be self-adjoint,  $\sigma(A) \subset \mathbb{R}$  the (non-empty, compact) spectrum of  $A$ . Finally, let  $R > \|A\|_2$  and  $g : U_R(0) \rightarrow \mathbb{C}$  be holomorphic such that  $g(0) = 0$ . Then*

$$(g|_{\sigma(A)})(A) \in \mathcal{I}_2.$$

*Proof.* Since  $R > \|A\|_2 \geq \|A\|$  and  $g(0) = 0$ , an application of Lemma 15 gives

$$(g|_{\sigma(A)})(A) = \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \cdot A^k.$$

Also, since  $g : U_R(0) \rightarrow \mathbb{C}$  is holomorphic and  $\|A\|_2 < R$ , the sequence

$$\left( \frac{g^{(k)}(0)}{k!} \cdot \|A\|_2^k \right)_{k \in \mathbb{N}^*}$$

is absolutely summable. Using that  $\mathcal{I}_2$  is a  $*$ -ideal in  $L(X, X)$  and that for all  $B, C \in \mathcal{I}_2$

$$\|B \circ C\|_2 \leq \|B\|_2 \cdot \|C\|_2,$$

see e.g., [53, Vol. II, Prop. 5, p. 41], it follows for every non-empty finite subset  $J \subset \mathbb{N}^*$ ,

$$\sum_{k \in J} \left\| \frac{g^{(k)}(0)}{k!} \cdot A^k \right\|_2 \leq \sum_{k \in J} \frac{|g^{(k)}(0)|}{k!} \cdot \|A\|_2^k \leq \sum_{k=1}^{\infty} \frac{|g^{(k)}(0)|}{k!} \cdot \|A\|_2^k.$$

As a consequence,

$$\left( \frac{g^{(k)}(0)}{k!} \cdot A^k \right)_{k \in \mathbb{N}^*}$$

is absolutely summable in  $(\mathcal{I}_2, \|\cdot\|_2)$ . Finally, since  $\mathcal{I}_2 \hookrightarrow L(X, X)$  is continuous, we conclude that

$$(g|_{\sigma(A)})(A) \in \mathcal{I}_2 .$$

□

**Remark 17.** *The same statement holds true for  $\mathcal{I}_1$  which denotes complex Banach space of the trace class operators on  $X$  and  $\|\cdot\|_1$  is the corresponding trace norm. The proof is virtually identical to the previous one.*

As a consequence of Lemma 16, we observe that  $\cos(t\sqrt{A})$  is a perturbation of the identity operator by a Hilbert-Schmidt operator. Likewise,  $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$  is a perturbation of a multiple of the identity operator by a Hilbert-Schmidt operator. The discussion so far involved the sum of two operators, a multiple of the identity and a convolution type operator, i.e.,  $c - C$ . More generally, applying similar methods used for a proof in the companion paper [12, Thm. 4.3], the following theorem gives that  $\cos(t\sqrt{A+B})$  is a perturbation of  $\cos(t\sqrt{A})$  by a Hilbert-Schmidt operator if  $B$  is Hilbert-Schmidt for  $t \in \mathbb{R}$ . Likewise,  $\frac{\sin(t\sqrt{A+B})}{\sqrt{A+B}}$  is a perturbation of  $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$  by a Hilbert-Schmidt operator if  $B$  is Hilbert-Schmidt for  $t \in \mathbb{R}$ .

The proof of the following theorem utilizes expansion of solution operators in terms of generalized hypergeometric functions given in the companion paper [12, Thm. 4.3].

**Theorem 18.** *Let  $(X, \langle \cdot | \cdot \rangle)$  be a non-trivial complex Hilbert space,  $\sqrt{\cdot}$  the complex square-root function, with domain  $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$ .  $A, B \in L(X, X)$  self-adjoint such that  $[A, B] = 0$  and  $\sigma(A), \sigma(A+B) \subset \mathbb{R}$  the (non-empty, compact) spectra of  $A$  and  $A+B$ , respectively. Let  $B \in \mathcal{I}_2$ , then the operators*

$$\begin{aligned} & \left[ \overline{\cos(t\sqrt{\cdot})} \Big|_{\sigma(A+B)} \right] (A+B) - \left[ \overline{\cos(t\sqrt{\cdot})} \Big|_{\sigma(A)} \right] (A) , \\ & \left[ \overline{\frac{\sin(t\sqrt{\cdot})}{\sqrt{\cdot}}} \Big|_{\sigma(A+B)} \right] (A+B) - \left[ \overline{\frac{\sin(t\sqrt{\cdot})}{\sqrt{\cdot}}} \Big|_{\sigma(A)} \right] (A) \end{aligned}$$

are elements of  $\mathcal{I}_2$ .

*Proof.* For the proof, we use that  $\mathcal{I}_2$  is a  $*$ -ideal in  $L(X, X)$  and that for all  $B \in L(X, X)$ ,  $C \in \mathcal{I}_2$

$$\|B \circ C\|_2 \leq \|B\| \cdot \|C\|_2 ,$$

see e.g., [53, Vol. II, Prop. 5, p. 41]. In the following,  ${}_0F_1$  denotes the generalized hypergeometric function, defined as in [48]. In a first step, we note for every  $k \in \mathbb{N}$ ,  $z \in \mathbb{C}$  that

$$\begin{aligned} \left| {}_0F_1\left(-; k + \frac{1}{2}; z\right) \right| &= \left| \sum_{l=0}^{\infty} \frac{z^l}{(k + \frac{1}{2})_l \cdot l!} \right| \leq \sum_{l=0}^{\infty} \frac{|z|^l}{(k + \frac{1}{2})_l \cdot l!} \leq \sum_{l=0}^{\infty} \frac{|z|^l}{(\frac{1}{2})^l \cdot l!} \\ &= \sum_{l=0}^{\infty} \frac{|2z|^l}{l!} = e^{2|z|} , \\ \left| {}_0F_1\left(-; k + \frac{3}{2}; z\right) \right| &= \left| \sum_{l=0}^{\infty} \frac{z^l}{(k + \frac{3}{2})_l \cdot l!} \right| \leq \sum_{l=0}^{\infty} \frac{|z|^l}{(k + \frac{3}{2})_l \cdot l!} \leq \sum_{l=0}^{\infty} \frac{|z|^l}{(\frac{3}{2})^l \cdot l!} \\ &= \sum_{l=0}^{\infty} \frac{|2z/3|^l}{l!} = e^{2|z|/3} , \end{aligned}$$

and hence, if in addition  $k > 0$ , for  $t \in \mathbb{R}$  that

$$\left\| (-1)^k \cdot \frac{t^{2k}}{(2k)!} \cdot \left\{ \left[ {}_0F_1\left(-; k + \frac{1}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right] (A) \right\} B^k \right\|_2$$

$$\begin{aligned}
&\leq e^{t^2\|A\|/2} \frac{|t|^{2k}}{(2k)!} \|B\|_2^k, \\
&\left\| (-1)^k \cdot \frac{t^{2k+1}}{(2k+1)!} \cdot \left\{ \left[ {}_0F_1\left(-; k + \frac{3}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right](A) \right\} B^k \right\|_2 \\
&\leq e^{3t^2\|A\|/2} \frac{|t|^{2k+1}}{(2k+1)!} \|B\|_2^k.
\end{aligned}$$

As a consequence, the sequences

$$\begin{aligned}
&\left( (-1)^k \cdot \frac{t^{2k}}{(2k)!} \cdot \left\{ \left[ {}_0F_1\left(-; k + \frac{1}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right](A) \right\} B^k \right)_{k \in \mathbb{N}^*}, \\
&\left( (-1)^k \cdot \frac{t^{2k+1}}{(2k+1)!} \cdot \left\{ \left[ {}_0F_1\left(-; k + \frac{3}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right](A) \right\} B^k \right)_{k \in \mathbb{N}^*}
\end{aligned}$$

are absolutely summable in  $\mathcal{I}_2$ . Since  $\|C\|_2 \geq \|C\|$  for every  $C \in \mathcal{I}_2$ , this implies that the operators

$$\begin{aligned}
&\left[ \overline{\cos(t\sqrt{\phantom{x}})} \Big|_{\sigma(A+B)} \right] (A+B) - \left[ {}_0F_1\left(-; \frac{1}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right](A), \\
&\left[ \frac{\overline{\sin(t\sqrt{\phantom{x}})}}{\sqrt{\phantom{x}}} \Big|_{\sigma(A+B)} \right] (A+B) - \left[ {}_0F_1\left(-; \frac{3}{2}; -\frac{t^2}{4} \cdot \text{id}_{\sigma(A)}\right) \right](A)
\end{aligned}$$

are elements of  $\mathcal{I}_2$ . From this, the statement follows with the help of [12, Lemma 4.4].  $\square$

In particular,  $\cos(t\sqrt{c-C})$  is a perturbation of  $\cos(t\sqrt{c})$  by a Hilbert-Schmidt operator if  $C$  is Hilbert-Schmidt for  $t \in \mathbb{R}$ . Likewise,  $\frac{\sin(t\sqrt{c-C})}{\sqrt{c-C}}$  is a perturbation of  $\frac{\sin(t\sqrt{c})}{\sqrt{c}}$  by a Hilbert-Schmidt operator if  $C$  is Hilbert-Schmidt for  $t \in \mathbb{R}$ . From the functional calculus for bounded, linear, self-adjoint operators on Hilbert spaces, it is easy to conclude that functions of  $c - C$  are functions of  $C$  in the following obvious way.

**Corollary 19.** *Let  $C \in \mathcal{I}_2$  and  $c > 0$ . In addition, let  $A = c$  and  $B = -C$ , then for every  $t \in \mathbb{R}$ , the operators*

$$\begin{aligned}
&\left[ \overline{\cos(t\sqrt{\phantom{x}})} \Big|_{\sigma(c-C)} \right] (c-C) - \cos(\sqrt{c}t) \\
&= \left[ \overline{\cos(t\sqrt{\phantom{x}})} \circ (c - \text{id}_{\sigma(C)}) - \cos(\sqrt{c}t) \right](C), \\
&\left[ \frac{\overline{\sin(t\sqrt{\phantom{x}})}}{\sqrt{\phantom{x}}} \Big|_{\sigma(c-C)} \right] (c-C) - \frac{\sin(\sqrt{c}t)}{\sqrt{c}} \\
&= \left[ \frac{\overline{\sin(t\sqrt{\phantom{x}})}}{\sqrt{\phantom{x}}} \circ (c - \text{id}_{\sigma(C)}) - \frac{\sin(\sqrt{c}t)}{\sqrt{c}} \right](C),
\end{aligned}$$

are elements of  $\mathcal{I}_2$ , where we define  $\sin 0/0 := 1$ .

*Proof.* The statement is an immediate consequence of Theorem 18.  $\square$

The previous results enable the treatment of the solutions of the homogeneous equation with periodic, antiperiodic, and Dirichlet BCs. In order to treat inhomogeneous equation and BCs that include derivatives such the Neumann BCs, we need more detailed information on the eigenvalues of the operators in Corollary 19, as will be given in Theorem 21.

**Lemma 20.** *Let  $c > 0$  and  $\lambda \leq \min\{c, 1\}$ . Then for every  $t \in \mathbb{R}$*

$$\begin{aligned}
&|\cos(t\sqrt{c-\lambda}) - \cos(t\sqrt{c})| \leq \left( \frac{t^2}{2c} + \frac{|t|}{\sqrt{c}} \right) |\lambda|, \\
&\left| \frac{\sin(t\sqrt{c-\lambda})}{\sqrt{c-\lambda}} - \frac{\sin(t\sqrt{c})}{\sqrt{c}} \right| \leq \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) |\lambda|.
\end{aligned}$$

*Proof.* For  $\lambda < c$ , we conclude that

$$\begin{aligned}
\cos(t\sqrt{c-\lambda}) - \cos(t\sqrt{c}) &= \cos(t[\sqrt{c-\lambda} - \sqrt{c}] + t\sqrt{c}) - \cos(t\sqrt{c}) \\
&= \cos(t[\sqrt{c-\lambda} - \sqrt{c}]) \cos(t\sqrt{c}) - \sin(t[\sqrt{c-\lambda} - \sqrt{c}]) \sin(t\sqrt{c}) - \cos(t\sqrt{c}) \\
&= \{\cos(t[\sqrt{c-\lambda} - \sqrt{c}]) - 1\} \cos(t\sqrt{c}) - \sin(t[\sqrt{c-\lambda} - \sqrt{c}]) \sin(t\sqrt{c}) \\
&= -2 \sin^2\left(\frac{t}{2}[\sqrt{c-\lambda} - \sqrt{c}]\right) \cos(t\sqrt{c}) - \sin(t[\sqrt{c-\lambda} - \sqrt{c}]) \sin(t\sqrt{c})
\end{aligned}$$

and hence

$$\begin{aligned}
|\cos(t\sqrt{c-\lambda}) - \cos(t\sqrt{c})| &\leq \frac{t^2}{2} [\sqrt{c-\lambda} - \sqrt{c}]^2 + |t| \cdot |\sqrt{c-\lambda} - \sqrt{c}| \\
&= \frac{t^2}{2} \left[ \frac{-\lambda}{\sqrt{c-\lambda} + \sqrt{c}} \right]^2 + |t| \cdot \left| \frac{-\lambda}{\sqrt{c-\lambda} + \sqrt{c}} \right| \leq \frac{t^2 \lambda^2}{2c} + \frac{|t| \cdot |\lambda|}{\sqrt{c}}.
\end{aligned}$$

Furthermore, using that for  $x, y > 0$

$$\begin{aligned}
\frac{\sin(x)}{x} - \frac{\sin(y)}{y} &= \int_0^1 [\cos(xu) - \cos(yu)] du \\
&= \int_0^1 [\cos((x-y)u + yu) - \cos(yu)] du \\
&= \int_0^1 \{[\cos((x-y)u) - 1] \cos(yu) + \sin((x-y)u) \sin(yu)\} du \\
&= \int_0^1 \left[ -2 \sin^2\left(\frac{x-y}{2}u\right) \cos(yu) + \sin((x-y)u) \sin(yu) \right] du
\end{aligned}$$

and hence that

$$\begin{aligned}
\left| \frac{\sin(x)}{x} - \frac{\sin(y)}{y} \right| &\leq \int_0^1 \left| -2 \sin^2\left(\frac{x-y}{2}u\right) \cos(yu) + \sin((x-y)u) \sin(yu) \right| du \\
&\leq \frac{|x-y|^2}{2} \int_0^1 u^2 du + |x-y| \int_0^1 u du = \frac{|x-y|^2}{6} + \frac{|x-y|}{2},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\left| \frac{\sin(t\sqrt{c-\lambda})}{\sqrt{c-\lambda}} - \frac{\sin(t\sqrt{c})}{\sqrt{c}} \right| &\leq \frac{t^2 |\sqrt{c-\lambda} - \sqrt{c}|^2}{6} + \frac{|t| \cdot |\sqrt{c-\lambda} - \sqrt{c}|}{2} \\
&= \frac{t^2}{6} \left[ \frac{-\lambda}{\sqrt{c-\lambda} + \sqrt{c}} \right]^2 + \frac{|t|}{2} \left| \frac{-\lambda}{\sqrt{c-\lambda} + \sqrt{c}} \right| \leq \frac{t^2 \lambda^2}{6c} + \frac{|t| \cdot |\lambda|}{2\sqrt{c}}
\end{aligned}$$

□

**Theorem 21.** Let  $(X, \langle \cdot | \cdot \rangle)$  be a non-trivial complex Hilbert space,  $\sqrt{\cdot}$  the complex square-root function, with domain  $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$ ,  $c > 0$  and  $t \in \mathbb{R}$ . Furthermore, let  $C \in L(X, X)$  be Hilbert-Schmidt,

$$(e_k)_{k \in \mathbb{N}^*}$$

a corresponding basis of eigenvectors and, for every  $k \in \mathbb{N}^*$ ,  $\lambda_k$  the eigenvalue of  $C$  that corresponds to  $e_k$ .

(i) Then

$$\lim_{k \rightarrow \infty} \lambda_k = 0 ;$$

(ii) if  $N \in \mathbb{N}^*$  is such that  $\lambda_k \leq \min\{c, 1\}$  for every  $k \in \mathbb{N}^*$  satisfying  $k \geq N$ , then

$$\begin{aligned}
\left| \left[ \overline{\cos(t\sqrt{\cdot})} \circ (c - id_{\sigma(C)}) - \cos(\sqrt{c}t) \right] (\lambda_k) \right| &\leq \left( \frac{t^2}{2c} + \frac{|t|}{\sqrt{c}} \right) |\lambda_k|, \\
\left| \left[ \frac{\overline{\sin(t\sqrt{\cdot})}}{\sqrt{\cdot}} \circ (c - id_{\sigma(C)}) - \frac{\sin(\sqrt{c}t)}{\sqrt{c}} \right] (\lambda_k) \right| &\leq \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) |\lambda_k|,
\end{aligned}$$

for every  $k \in \mathbb{N}^*$  satisfying  $k \geq N$ .

*Proof.* Part (i): First, we note that

$$C e_k = \lambda_k \cdot e_k, \quad C \xi = C \sum_{k=1}^{\infty} \langle e_k | \xi \rangle_2 \cdot e_k = \sum_{k=1}^{\infty} \langle e_k | \xi \rangle_2 \cdot C e_k = \sum_{k=1}^{\infty} \lambda_k \langle e_k | \xi \rangle_2 \cdot e_k,$$

for every  $\xi \in X$ . In particular, since for every  $k \in \mathbb{N}^*$

$$\|C e_k\|_2^2 = \|C e_k\|_2^2 = |\lambda_k|^2$$

and  $C$  is Hilbert-Schmidt,

$$(|\lambda_k|^2)_{k \in \mathbb{N}^*}$$

is summable. As a consequence,

$$\lim_{k \rightarrow \infty} \lambda_k = 0.$$

The statement of Part (ii) is a direct consequence of Lemma 20.  $\square$

**Remark 22.** Note that the proof of Theorem 21, together with an application of the spectral theorem for bounded self-adjoint operators on Hilbert spaces, provides an independent proof of Corollary 19. In addition, Theorem 21 provides the basis for the application of Corollary 24.

### 3.3 Satisfying Boundary Conditions not Involving Derivatives

Now we are in a position to study BCs not involving derivatives for operators with a pure point spectrum. The Hilbert-Schmidt property leads to a uniform convergence argument which allows us to interchange limits. Hence, BCs are automatically satisfied. In addition, the Hilbert-Schmidt property leads to *smoothing* of the input, in the sense that an  $L^2$  function is mapped into a function that is continuous up to the boundary.

**Theorem 23. (Smoothing Functions of an Operator)** Let  $c, K > 0$ ,  $n \in \mathbb{N}^*$ ,  $\Omega \subset \mathbb{R}^n$  be non-empty, bounded and open,  $A$  be a densely-defined, linear and self-adjoint operator in  $L^2_{\mathbb{C}}(\Omega)$  with a pure point spectrum  $\sigma(A)$ , i.e., for which there is a Hilbert basis

$$(e_k)_{k \in \mathbb{N}^*}$$

of eigenvectors. In particular, for every  $k \in \mathbb{N}^*$ , let  $\lambda_k$  be the eigenvalue corresponding to  $e_k$ . Furthermore, let  $\hat{\Omega} \supset \bar{\Omega}$  be bounded and open, and for every  $k \in \mathbb{N}^*$  let  $e_k$  be the restriction of some  $\hat{e}_k \in C(\hat{\Omega}, \mathbb{C})$  satisfying

$$\|\hat{e}_k\|_{\infty} \leq K.$$

Finally, let  $f \in U^s_{\mathbb{C}}(\sigma(A), \mathbb{C})$ ,  $u \in L^2_{\mathbb{C}}(\Omega)$  and  $b : \mathbb{R} \rightarrow L^2_{\mathbb{C}}(\Omega)$  be continuous.

(i)  $f(A)$  is Hilbert-Schmidt if and only if

$$(|f(\lambda_k)|^2)_{k \in \mathbb{N}^*}$$

is summable.

(ii) If  $f(A)$  is Hilbert-Schmidt, then  $f(A)u$  has an extension to a continuous function on  $\bar{\Omega}$ , and for every limit point  $x$  of  $\Omega$

$$\lim_{y \rightarrow x} [f(A)u](y) = \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | u \rangle \left( \lim_{y \rightarrow x} e_k(y) \right).$$

(iii) If in addition,  $f$  is real-valued such that  $f(A)$  is Hilbert-Schmidt and  $f(A) \leq c$ ,

a) then  $v : \mathbb{R} \rightarrow X$ , for every  $t \in \mathbb{R}$  defined by

$$v(t) := \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} \Big|_{\sigma(c-f(A))} \right] (c - f(A))b(\tau) d\tau,$$

where  $\int$  denotes weak integration in  $X$ ,

$$I_t := \begin{cases} [0, t] & \text{if } t \geq 0 \\ [t, 0] & \text{if } t < 0 \end{cases},$$

for every  $t \in \mathbb{R}$ , satisfies

$$v(t) = \sum_{k=1}^{\infty} \left\{ \int_{I_t} \frac{\overline{\sin((t-\tau)\sqrt{\cdot})}}{\sqrt{\cdot}} (c - f(\lambda_k)) \langle e_k | b(\tau) \rangle d\tau \right\} e_k.$$

b) then for every  $t \in \mathbb{R}$ ,  $v(t) - v_c(t)$  has an extension to a continuous function on  $\bar{\Omega}$ , and for every limit point  $x$  of  $\Omega$

$$\begin{aligned} & \lim_{y \rightarrow x} [v(t) - v_c(t)](y) \\ &= \sum_{k=1}^{\infty} \left\{ \int_{I_t} \left[ \frac{\overline{\sin((t-\tau)\sqrt{\cdot})}}{\sqrt{\cdot}} (c - f(\lambda_k)) \right. \right. \\ & \quad \left. \left. - \frac{\overline{\sin((t-\tau)\sqrt{\cdot})}}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \left( \lim_{y \rightarrow x} e_k(y) \right). \end{aligned}$$

where,  $v_c : \mathbb{R} \rightarrow L^2_{\mathbb{C}}(\Omega)$  is defined by

$$v_c(t) := \int_{I_t} \frac{\overline{\sin((t-\tau)\sqrt{\cdot})}}{\sqrt{\cdot}} (c) \cdot b(\tau) d\tau.$$

*Proof.* Part (i): From the spectral theorem for densely-defined, linear and self-adjoint Hilbert spaces, it follows for every  $k \in \mathbb{N}^*$  and  $h \in L^2_{\mathbb{C}}(\Omega)$  that

$$\begin{aligned} f(A) e_k &= f(\lambda_k) \cdot e_k, \\ f(A) h &= f(A) \sum_{k=1}^{\infty} \langle e_k | h \rangle_2 \cdot e_k = \sum_{k=1}^{\infty} \langle e_k | h \rangle_2 \cdot f(A) e_k = \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | h \rangle_2 \cdot e_k. \end{aligned}$$

Hence  $f(A)$  has a pure point spectrum, and its spectrum  $\sigma(f(A))$  is given by

$$\sigma(f(A)) = \overline{\{f(\lambda_k) : k \in \mathbb{N}^*\}}.$$

In particular, since for every  $k \in \mathbb{N}^*$

$$\|f(A) e_k\|_2^2 = \|f(\lambda_k) e_k\|_2^2 = |f(\lambda_k)|^2,$$

$f(A)$  is Hilbert-Schmidt if and only if

$$(|f(\lambda_k)|^2)_{k \in \mathbb{N}^*}$$

is summable.

Part (ii): It follows for  $m, m' \in \mathbb{N}^*$  satisfying  $N \leq m \leq m'$ , where  $N \in \mathbb{N}^*$  is sufficiently large, that

$$\begin{aligned} & \left\| \sum_{k=m}^{m'} f(\lambda_k) \langle e_k | u \rangle_2 \hat{e}_k \right\|_{\infty} \leq K \sum_{k=m}^{m'} |f(\lambda_k)| \cdot |\langle e_k | u \rangle_2| \\ & \leq K \cdot \left( \sum_{k=m}^{m'} |f(\lambda_k)|^2 \right)^{1/2} \cdot \left( \sum_{k=m}^{m'} |\langle e_k | u \rangle_2|^2 \right)^{1/2} \leq K \cdot \|u\|_2 \cdot \left( \sum_{k=m}^{m'} |f(\lambda_k)|^2 \right)^{1/2} \\ & \left( \leq K \cdot \|u\|_2 \cdot \left( \sum_{k=1}^{\infty} |f(\lambda_k)|^2 \right)^{1/2} \right). \end{aligned}$$



As a consequence,

$$\left( \sum_{k=1}^N f(\lambda_k) \langle e_k | u \rangle_2 \hat{e}_k \right)_{N \in \mathbb{N}^*}$$

is a Cauchy sequence in  $(BC(\hat{\Omega}, \mathbb{C}), \|\cdot\|_\infty)$ , (i.e., bounded continuous functions defined on  $\hat{\Omega}$ ) and hence uniformly convergent to an extension of  $f(A)u$  to a bounded continuous function on  $\hat{\Omega}$ . In particular, this implies that for every limit point  $x$  of  $\Omega$  that

$$\begin{aligned} & \lim_{y \rightarrow x} [f(A)u](y) \\ &= \lim_{y \rightarrow x} \lim_{N \rightarrow \infty} \sum_{k=1}^N f(\lambda_k) \langle e_k | u \rangle \hat{e}_k(y) = \lim_{N \rightarrow \infty} \lim_{y \rightarrow x} \sum_{k=1}^N f(\lambda_k) \langle e_k | u \rangle \hat{e}_k(y) \\ &= \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | u \rangle \left( \lim_{y \rightarrow x} e_k(y) \right). \end{aligned}$$

For the latter, see, e.g., in [11, Thm 2.41].

Part (iii)a): First, for all  $t \in \mathbb{R}$  and every  $k \in \mathbb{N}^*$ , it follows from the spectral theorem for densely-defined, linear and self-adjoint in Hilbert spaces that

$$\begin{aligned} \langle e_k | v(t) \rangle &= \int_{I_t} \langle e_k | \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} \Big|_{\sigma(c-f(A))} \right] (c-f(A)) b(\tau) \rangle d\tau \\ &= \int_{I_t} \left\langle \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} \Big|_{\sigma(c-f(A))} \right] (c-f(A)) e_k | b(\tau) \right\rangle d\tau \\ &= \int_{I_t} \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c-f(\lambda_k)) \langle e_k | b(\tau) \rangle d\tau \end{aligned}$$

and hence that

$$v(t) = \sum_{k=1}^{\infty} \left\{ \int_{I_t} \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c-f(\lambda_k)) \langle e_k | b(\tau) \rangle d\tau \right\} e_k.$$

Part(iii)b): If  $k \in \mathbb{N}^*$  is such that  $f(\lambda_k) \leq \min\{c, 1\}$ , then

$$\left| \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c-f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right| \leq \left[ \frac{(t-\tau)^2}{6c} + \frac{|t-\tau|}{2\sqrt{c}} \right] \cdot |f(\lambda_k)|$$

and hence

$$\begin{aligned} & \left| \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c-f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right| \\ & \leq \int_{I_t} \left[ \frac{(t-\tau)^2}{6c} + \frac{|t-\tau|}{2\sqrt{c}} \right] \cdot |f(\lambda_k)| \cdot |\langle e_k | b(\tau) \rangle| d\tau \\ & \leq \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) |f(\lambda_k)| \int_{I_t} |\langle e_k | b(\tau) \rangle| d\tau \end{aligned}$$

It follows for  $m, m' \in \mathbb{N}^*$  satisfying  $N \leq m \leq m'$ , where  $N \in \mathbb{N}^*$  is sufficiently large, that

$$\begin{aligned} & \left\| \sum_{k=m}^{m'} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c-f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}_k \right\|_{\infty} \\ & \leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \int_{I_t} \left[ \sum_{k=m}^{m'} |f(\lambda_k)| \cdot |\langle e_k | b(\tau) \rangle| \right] d\tau \\ & \leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \cdot \left[ \sum_{k=m}^{m'} |f(\lambda_k)|^2 \right]^{1/2} \cdot \int_{I_t} \left[ \sum_{k=m}^{m'} |\langle e_k | b(\tau) \rangle|^2 \right]^{1/2} d\tau \end{aligned}$$

$$\leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \cdot \left[ \sum_{k=m}^{m'} |f(\lambda_k)|^2 \right]^{1/2} \cdot \int_{I_t} \|b(\tau)\|_2 d\tau .$$

As a consequence,

$$\left( \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}_k \right)_{N \in \mathbb{N}^*}$$

is a Cauchy sequence in  $(BC(\hat{\Omega}, \mathbb{C}), \|\cdot\|_\infty)$  and hence uniformly convergent to an extension of  $f(A)g$  to a bounded continuous function on  $\hat{\Omega}$ . In particular, this implies that for every limit point  $x$  of  $\Omega$  that

$$\begin{aligned} & \lim_{y \rightarrow x} [v(t) - v_c(t)](y) \\ &= \lim_{y \rightarrow x} \lim_{N \rightarrow \infty} \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} e_k(y) \\ &= \lim_{N \rightarrow \infty} \lim_{y \rightarrow x} \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} e_k(y) \\ &= \sum_{k=1}^{\infty} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \left( \lim_{y \rightarrow x} e_k(y) \right) . \end{aligned}$$

For the latter, see, e.g., [11, Thm 2.41]. □

As a consequence, for the example of the Dirichlet BCs in Section 4.4, solutions to the wave equation corresponding to data  $u(0, \cdot), u'(0, \cdot) \in L^2_{\mathbb{C}}(I)$  satisfying pointwise for  $x$  in a some neighborhood of  $-1$  and  $1$

$$\lim_{x \rightarrow -1} u(x, 0) = \lim_{x \rightarrow 1} u(x, 0) = 0 ,$$

in the same sense, satisfy the Dirichlet BCs for all  $t \in \mathbb{R}$ . In addition, to the micromoduli considered in Section 4.3, Corollary 24 is applicable.

### 3.4 Satisfying Boundary Conditions Involving Derivatives

Now we are in a position to study BCs involving derivatives for operators with a pure point spectrum. With additional decay conditions of the eigenvalues of  $f(A)$ , we reach a uniform convergence argument also for derivatives which allows us to interchange limits. As a consequence, BCs are automatically satisfied. For instance, solutions to the wave equation in Section 4.3, for data  $u(0, \cdot), u'(0, \cdot) \in L^2_{\mathbb{C}}(I)$  satisfying pointwise for  $x$  in a some neighborhood of  $-1$  and  $1$

$$\lim_{x \rightarrow -1} u'(x, 0) = \lim_{x \rightarrow 1} u'(x, 0) = 0 ,$$

in the same sense, satisfy the Neumann BCs for all  $t \in \mathbb{R}$ .

**Corollary 24. (Smoothing Functions of an Operator II)** *In addition to the assumptions of Theorem 23, we assume that  $\Omega = I$ ,  $\hat{\Omega} = \hat{I}$  and  $I, \hat{I}$  are non-empty open intervals of  $\mathbb{R}$ . Furthermore,*

we assume that  $f(A)$  is Hilbert-Schmidt and for every  $k \in \mathbb{N}^*$  that  $e_k$  is differentiable with a derivative that has an extension  $\hat{e}'_k$  to a continuous function on  $\bar{I}$ . Finally, we assume that

$$(\| \hat{e}'_k \|_\infty f(\lambda_k))^2_{k \in \mathbb{N}^*}$$

is summable.

(i) Then  $f(A)u \in C^1(\bar{I}, \mathbb{C})$ , and for every limit point  $x$  of  $I$

$$\begin{aligned} \lim_{y \rightarrow x} [f(A)u](y) &= \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | u \rangle \left( \lim_{y \rightarrow x} e_k(y) \right), \\ \lim_{y \rightarrow x} [f(A)u]'(y) &= \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | u \rangle \left( \lim_{y \rightarrow x} \hat{e}'_k(y) \right). \end{aligned}$$

(ii) If in addition,  $f$  is real-valued such that  $f(A) \leq c$  and  $v, v_c$  are defined as in Theorem 23 (iii)a), then for every  $t \in \mathbb{R}$ ,  $v(t) - v_c(t) \in C^1(\bar{I}, \mathbb{C})$  and for every limit point  $x$  of  $\Omega$

$$\begin{aligned} &\lim_{y \rightarrow x} [v(t) - v_c(t)](y) \\ &= \sum_{k=1}^{\infty} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) \right. \right. \\ &\quad \left. \left. - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \left( \lim_{y \rightarrow x} e_k(y) \right), \\ &\lim_{y \rightarrow x} [v(t) - v_c(t)]'(y) \\ &= \sum_{k=1}^{\infty} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) \right. \right. \\ &\quad \left. \left. - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \left( \lim_{y \rightarrow x} \hat{e}'_k(y) \right). \end{aligned}$$

*Proof.* Part (i): It follows for  $m, m' \in \mathbb{N}^*$  satisfying  $N \leq m \leq m'$ , where  $N \in \mathbb{N}^*$  is sufficiently large, that

$$\begin{aligned} &\left\| \sum_{k=m}^{m'} f(\lambda_k) \langle e_k | u \rangle_2 \hat{e}'_k \right\|_\infty \leq \sum_{k=m}^{m'} |f(\lambda_k)| \|\hat{e}'_k\|_\infty \cdot |\langle e_k | u \rangle_2| \\ &\leq \cdot \left( \sum_{k=m}^{m'} \|\hat{e}'_k\|_\infty f(\lambda_k)^2 \right)^{1/2} \cdot \left( \sum_{k=m}^{m'} |\langle e_k | u \rangle_2|^2 \right)^{1/2} \\ &\leq \|u\|_2 \cdot \left( \sum_{k=m}^{m'} \|\hat{e}'_k\|_\infty f(\lambda_k)^2 \right)^{1/2} \left( \leq \cdot \|u\|_2 \cdot \left( \sum_{k=1}^{\infty} \|\hat{e}'_k\|_\infty f(\lambda_k)^2 \right)^{1/2} \right). \end{aligned}$$

As a consequence,

$$\left( \sum_{k=1}^N f(\lambda_k) \langle e_k | u \rangle_2 \hat{e}'_k \right)_{N \in \mathbb{N}^*}$$

is a Cauchy sequence in  $(C(\bar{I}, \mathbb{C}), \|\cdot\|_\infty)$  and hence uniformly convergent to a continuous function on  $\bar{I}$ . In particular, this implies that  $f(A)u$  is continuously differentiable and for every limit point  $x$  of  $I$  that

$$\lim_{y \rightarrow x} [f(A)u]'(y) = \sum_{k=1}^{\infty} f(\lambda_k) \langle e_k | u \rangle \left( \lim_{y \rightarrow x} \hat{e}'_k(y) \right).$$

For the latter, see, e.g., [11, Thm 2.42].

Part (ii): It follows for  $t \in \mathbb{R}$ ,  $m, m' \in \mathbb{N}^*$  satisfying  $N \leq m \leq m'$ , where  $N \in \mathbb{N}^*$  is sufficiently large, that

$$\begin{aligned}
& \left\| \sum_{k=m}^{m'} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}'_k \right\|_{\infty} \\
& \leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \int_{I_t} \left[ \sum_{k=m}^{m'} \|\hat{e}'_k\|_{\infty} |f(\lambda_k)| \cdot |\langle e_k | b(\tau) \rangle| \right] d\tau \\
& \leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \cdot \left[ \sum_{k=m}^{m'} \|\hat{e}'_k\|_{\infty} |f(\lambda_k)|^2 \right]^{1/2} \cdot \left[ \int_{I_t} \left[ \sum_{k=m}^{m'} |\langle e_k | b(\tau) \rangle|^2 \right] d\tau \right]^{1/2} \\
& \leq K \left( \frac{t^2}{6c} + \frac{|t|}{2\sqrt{c}} \right) \cdot \left[ \sum_{k=m}^{m'} \|\hat{e}'_k\|_{\infty} |f(\lambda_k)|^2 \right]^{1/2} \cdot \int_{I_t} \|b(\tau)\|_2 d\tau.
\end{aligned}$$

As a consequence,

$$\left( \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}'_k \right)_{N \in \mathbb{N}^*}$$

is a Cauchy sequence in  $(C(\tilde{I}, \mathbb{C}), \|\cdot\|_{\infty})$  and hence uniformly convergent to a continuous function on  $\tilde{I}$ . In particular, this implies that  $v(t) - v_c(t)$  is continuously differentiable and for every limit point  $x$  of  $I$  that

$$\begin{aligned}
& \lim_{y \rightarrow x} [v(t) - v_c(t)]'(y) \\
& = \lim_{y \rightarrow x} \lim_{N \rightarrow \infty} \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}'_k(y) \\
& = \lim_{N \rightarrow \infty} \lim_{y \rightarrow x} \sum_{k=1}^N \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \hat{e}'_k(y) \\
& = \sum_{k=1}^{\infty} \left\{ \int_{I_t} \left[ \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c - f(\lambda_k)) - \frac{\sin((t-\tau)\sqrt{\cdot})}{\sqrt{\cdot}} (c) \right] \langle e_k | b(\tau) \rangle d\tau \right\} \left( \lim_{y \rightarrow x} \hat{e}'_k(y) \right).
\end{aligned}$$

For the latter, see, e.g., [11, Thm 2.41]. □

## 4 Study of Convolutions with Various Boundary Conditions

We study one-dimensional elasticity, which is an instance of regular Sturm-Liouville theory with prominent BCs such as periodic, antiperiodic, Dirichlet, and Neumann. In regular Sturm-Liouville problems, all BCs leading to self-adjoint operators are known [64, Thm. 13.14]. If needed, all associated BCs can be considered. All regular Sturm-Liouville operators are known to have a purely discrete spectrum, in particular, there is a Hilbert basis of eigenfunctions. There are a number of standard problems in higher dimensions that can be reduced to regular Sturm-Liouville

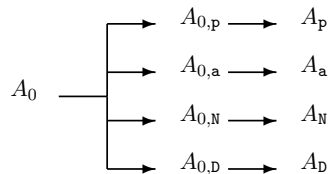


Figure 4.1: We extend the minimal operator  $A_0$ , specifying boundary conditions such as periodic, antiperiodic, Neumann, and Dirichlet boundary conditions, to an essentially self-adjoint operator  $A_{0,p}, A_{0,a}, A_{0,N}, A_{0,D}$ , respectively. Finally, we arrive at self-adjoint operators  $A_p, A_a, A_N, A_D$  by taking the closure of  $A_{0,p}, A_{0,a}, A_{0,N}, A_{0,D}$ , respectively.

problems on bounded domains. Also, generically, a differential operator with regular coefficients on  $\mathbb{R}^n$  has a purely discrete spectrum, providing an eigenbasis of the underlying space. Since the essential ingredient is a self-adjoint operator with a purely discrete spectrum, hence, our approach can easily cover higher spatial dimensions.

The choice of a Hilbert basis determines an abstract convolution, which we refer to as *canonical*. The most relevant BCs in applications are Dirichlet and Neumann BCs. In these cases, the connection of the abstract convolution to an integral form is not direct. On the other hand, for periodic and antiperiodic BCs, that connection is direct, needing a periodic and antiperiodic extension of the micromodulus function, respectively. Because of this directness, we choose to include periodic and antiperiodic BCs.

In the case of Neumann and Dirichlet BCs, we study additional convolutions that we refer to as “*simple*.” These are inspired by the convolutions from the periodic and antiperiodic BCs. Certain combinations of convolutions derived from periodic and antiperiodic BCs of even micromoduli with even and odd input function enforce Neumann and Dirichlet BCs in these simple convolutions. For instance, we sketch the case of Dirichlet BC. Let  $\hat{C}_p$  and  $\hat{C}_a$  denote periodic and antiperiodic extensions of  $C$ , respectively. It is easy to see that

$$C *_p u(1) = C *_p u(-1),$$

for any  $C$ . In addition, if  $C$  is even,  $C *_p u(1) = 0$  when  $u$  is odd. Likewise, for the antiperiodic case,

$$C *_a u(1) = -C *_a u(-1),$$

holds for any  $C$ . If  $C$  is even,  $C *_a u(1) = 0$  when  $u$  is even. This suggests that  $C *_p P_{\text{odd}}$  and  $C *_a P_{\text{even}}$  are functions of the classical operator. For the Neumann BC, the situation is similar. Namely,  $C *_p P_{\text{even}}$  and  $C *_a P_{\text{odd}}$  are functions of the classical operator as well, where  $P_{\text{odd}}$  and  $P_{\text{even}}$  are orthogonal projections onto odd and even functions, respectively. We elaborate on these examples in Sections 4.4.3 and 4.3.3, respectively.

Integral operators on bounded domains are often Hilbert-Schmidt, and hence, compact. Indeed, for all these BCs, we show that a simple decay condition on the regulating function leads to a Hilbert-Schmidt operator.

In Sections 4.3.5 and 4.4.6, we connect the eigenfunction expansions for the Neumann and Dirichlet BCs to that of (periodic) Fourier expansions on the extended domain  $(-2, 2)$ . This enables the application of standard results from Fourier theory to the particular eigenfunction expansions in these cases.

We define the minimal operator  $A_0 : C_0^2(I, \mathbb{C}) \rightarrow L_{\mathbb{C}}^2(I)$  by

$$A_0 u := -a_0 u'',$$

where  $a_0$  is a suitable real number and  $u \in C_0^2(I, \mathbb{C})$ . The operator  $A_0$  is densely defined, linear, and symmetric, but not essentially self-adjoint. We give self-adjoint extensions  $A_0$  by the closure of essentially self-adjoint operators. The extension process is depicted in Figure 4.

## 4.1 Periodic Boundary Conditions

We define the operator  $A_{0,p} : D(A_{0,p}) \rightarrow L_{\mathbb{C}}^2(I)$  by

$$D(A_{0,p}) := \left\{ u \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \rightarrow -1} u(x) = \lim_{x \rightarrow 1} u(x), \quad \lim_{x \rightarrow -1} u'(x) = \lim_{x \rightarrow 1} u'(x) \right\}$$

and

$$A_{0,p}u := -\frac{1}{\pi^2} u''$$

for every  $u \in D(A_{0,p})$ , where  $I := (-1, 1)$ ,  $C^2(\bar{I}, \mathbb{C})$  consists of the restrictions of the elements of  $C^2(J, \mathbb{C})$  to  $I$ , where  $J$  runs through all open intervals of  $\mathbb{R}$  containing  $\bar{I}$ . Note that  $C^2(\bar{I}, \mathbb{C})$  is a *dense subspace* of  $X$ .  $A_{0,p}$  is densely-defined, linear and symmetric.

#### 4.1.1 Associated Hilbert Basis and Properties

We note that  $A_{0,p}$  is a special case of a regular Sturm-Liouville operator. In particular,  $A_{0,p}$  is essentially self-adjoint. The closure  $A_p$  of  $A_{0,p}$  is given by

$$A_p u = -\frac{1}{\pi^2} u'',$$

where  $'$  denotes the weak derivative and  $u$  is a restriction to  $I$  of an periodic element of  $W^2(\mathbb{R}, \mathbb{C})$ .  $A_p$  has a purely discrete spectrum  $\sigma(A_p)$  consisting of the eigenvalues,

$$\sigma(A_p) = \{k^2 : k \in \mathbb{N}\}.$$

For every  $k \in \mathbb{Z}$ , a normalized eigenvector corresponding to the eigenvalue  $k^2$  is given by

$$e_k(x) := \frac{1}{\sqrt{2}} e^{i\pi k x}.$$

Hence,  $(e_k)_{k \in \mathbb{Z}}$  is a Hilbert basis of  $L^2_{\mathbb{C}}(I)$ , 0 is a simple eigenvalue and for every  $k \in \mathbb{N}^*$ ,  $k^2$  is an eigenvalue of geometric multiplicity 2, with corresponding linearly independent eigenvectors  $e_k, e_{-k}$ .

#### 4.1.2 Compactness of $f(A_p)$

For every  $f \in B(\sigma(A_p), \mathbb{C})$ , if

$$(|f(k^2)|^2)_{k \in \mathbb{N}}$$

is summable, then  $f(A_p)$  is a Hilbert-Schmidt operator and hence compact. The latter is the case if

$$|f(\lambda)| \leq c \lambda^{-\alpha}$$

for every  $\lambda \in \sigma(A_p)$ , where  $\alpha > 1/2$ ,  $c \geq 0$ .

#### 4.1.3 Properties of Canonical Convolutions and Integral Representations

In the following,  $\ast_p$  denotes the convolution in  $L^2_{\mathbb{C}}(I)$  that, according to Theorem 1, is associated to the Hilbert basis  $(e_k)_{k \in \mathbb{Z}}$ . In particular, for even  $C \in L^2(I)$ ,  $c \in \mathbb{R}$ ,

$$\langle e_k | C \rangle_2 = \langle e_{-k} | C \rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 \cos(\pi k y) \cdot C(y) dy, \quad k \in \mathbb{N}.$$

Hence,  $c - C \ast_p \cdot$  is a bounded self-adjoint function of  $A_p$ . Furthermore, if  $C$  is in addition positive and

$$c := \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy,$$

then  $c - C \ast_p \cdot$  is in addition positive, with a spectrum that contains 0.

In addition, for  $C, u \in L^2_{\mathbb{C}}(I)$

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= \frac{1}{2} \cdot e^{-i\pi k x} \cdot \int_{-1}^1 C^*(y) e^{i\pi k y} dy = \frac{1}{2} \cdot \int_{-1}^1 C^*(y) e^{i\pi k(y-x)} dy \\ &= \int_{-1+x}^{1+x} \hat{C}_p^*(y) e^{i\pi k(y-x)} dy = \frac{1}{2} \cdot \int_{-1}^1 \hat{C}_p^*(y+x) \cdot e^{i\pi k y} dy \\ &= \frac{1}{2} \cdot \int_{-1}^1 e^{-i\pi k y} \cdot \hat{C}_p^*(x-y) dy = \frac{1}{\sqrt{2}} \langle e_k | \hat{C}_p^*(x - \cdot) \rangle_2, \end{aligned}$$

where  $\hat{C}_p$  denotes the extension of  $C$  to a 2-periodic function on  $\mathbb{R}$ . Since for every finite subset  $S \subset \mathbb{N}$ ,

$$\sum_{k \in S} |(e_k(x))^* \cdot \langle C|e_k \rangle_2|^2 \leq \sum_{k \in S} |\langle e_k|C \rangle_2|^2 \leq \sum_{k \in \mathbb{N}} |\langle e_k|C \rangle_2|^2,$$

$(|(e_k(x))^* \cdot \langle C|e_k \rangle_2|^2)_{k \in \mathbb{N}}$  is summable.

Hence, we note that

$$\begin{aligned} (C *_p u)(x) &= \sum_{l \in \mathbb{N}} \langle e_{\beta(l)}|C \rangle_2 \langle e_{\beta(l)}|u \rangle_2 \cdot e_{\beta(l)}(x) = \langle \sum_{l \in \mathbb{N}} (e_{\beta(l)}(x))^* \cdot \langle C|e_{\beta(l)} \rangle_2 \cdot e_{\beta(l)}|u \rangle_2 \\ &= \langle \sum_{l \in \mathbb{N}} \frac{1}{\sqrt{2}} \langle e_{\beta(l)}|\hat{C}_p^*(x - \cdot) \rangle_2 \cdot e_{\beta(l)}|u \rangle_2 = \frac{1}{\sqrt{2}} \langle \sum_{l \in \mathbb{N}} \langle e_{\beta(l)}|\hat{C}_p^*(x - \cdot) \rangle_2 \cdot e_{\beta(l)}|u \rangle_2 \\ &= \frac{1}{\sqrt{2}} \langle \hat{C}_p^*(x - \cdot)|u \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \hat{C}_p(x - y) \cdot u(y) dy, \end{aligned}$$

where  $\beta : \mathbb{N} \rightarrow \mathbb{Z}$  is some bijection.

## 4.2 Antiperiodic Boundary Conditions

We define the operator  $A_{0,a} : D(A_{0,a}) \rightarrow L^2_{\mathbb{C}}(I)$  by

$$D(A_{0,a}) := \left\{ u \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \rightarrow -1} u(x) = -\lim_{x \rightarrow 1} u(x), \lim_{x \rightarrow -1} u'(x) = -\lim_{x \rightarrow 1} u'(x) \right\}$$

and

$$A_{0,a}u := -\frac{1}{\pi^2} u''$$

for every  $u \in D(A_{0,a})$ , where  $I := (-1, 1)$ ,  $C^2(\bar{I}, \mathbb{C})$  consists of the restrictions of the elements of  $C^2(J, \mathbb{C})$  to  $I$ , where  $J$  runs through all open intervals of  $\mathbb{R}$  containing  $\bar{I}$ . Note that  $C^2(\bar{I}, \mathbb{C})$  is a *dense subspace* of  $X$ .  $A_{0,a}$  is densely-defined, linear and symmetric.

### 4.2.1 Associated Hilbert Basis and Properties

We note that  $A_{0,a}$  is a special case of a regular Sturm-Liouville operator. In particular,  $A_{0,a}$  is essentially self-adjoint. The closure  $A_a$  of  $A_{0,a}$  is given by

$$A_a u = -\frac{1}{\pi^2} u'',$$

where  $\prime$  denotes the weak derivative and  $u$  is a restriction to  $I$  of an antiperiodic element of  $W^2(\mathbb{R}, \mathbb{C})$ .  $A_a$  has a purely discrete spectrum  $\sigma(A_a)$  consisting of the eigenvalues,

$$\sigma(A_a) = \left\{ \left( k + \frac{1}{2} \right)^2 : k \in \mathbb{N} \right\}.$$

For every  $k \in \mathbb{Z}$ , a normalized eigenvector corresponding to the eigenvalue  $(k + \frac{1}{2})^2$  is given by

$$e_k(x) := \frac{1}{\sqrt{2}} e^{i\pi(k + \frac{1}{2})x}$$

Hence  $(e_k)_{k \in \mathbb{Z}}$  is a Hilbert basis of  $L^2_{\mathbb{C}}(I)$ , and for every  $k \in \mathbb{N}$ ,  $(k + (1/2))^2$  is an eigenvalue of geometric multiplicity 2, with corresponding linearly independent eigenvalues  $e_k, e_{-k-1}$ .

### 4.2.2 Compactness of $f(A_a)$

For every  $f \in B(\sigma(A_a), \mathbb{C})$ , if

$$\left( \left| f \left( \left[ k + \frac{1}{2} \right]^2 \right) \right|^2 \right)_{k \in \mathbb{Z}}$$

is summable, then  $f(A_a)$  is a Hilbert-Schmidt operator and hence compact. In particular, the latter is the case if

$$|f(\lambda)| \leq c \lambda^{-\alpha}$$

for every  $\lambda \in \sigma(A_a)$ , where  $\alpha > 1/2$ ,  $c \geq 0$ .

### 4.2.3 Properties of Canonical Convolutions and Integral Representations

In the following,  $\ast_{\mathbf{a}}$  denotes the convolution in  $L^2_{\mathbb{C}}(I)$  that, according to Theorem 1, is associated to the Hilbert basis  $(e_k)_{k \in \mathbb{Z}}$ . In particular, for even  $C \in L^2(I)$  and  $c \in \mathbb{R}$ ,

$$\langle e_k | C \rangle_2 = \langle e_{-k-1} | C \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \cos \left[ \pi \left( k + \frac{1}{2} \right) y \right] \cdot C(y) dy, \quad k \in \mathbb{N}.$$

Hence all members of the sequence  $(\langle e_k | C \rangle_2)_{k \in \mathbb{Z}}$  are real-valued. Therefore  $c - C \ast_{\mathbf{a}} \cdot$  is a self-adjoint bounded function of  $A_{\mathbf{a}}$ . Furthermore, if  $C$  is in addition positive and

$$c := \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy,$$

then  $c - C \ast_{\mathbf{a}} \cdot$  is in addition positive.

In addition, for  $C, u \in L^2_{\mathbb{C}}(I)$

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= \frac{1}{2} \cdot e^{-i\pi(k+\frac{1}{2})x} \cdot \int_{-1}^1 C^*(y) e^{i\pi(k+\frac{1}{2})y} dy = \frac{1}{2} \cdot \int_{-1}^1 C^*(y) e^{i\pi(k+\frac{1}{2})(y-x)} dy \\ &= \int_{-1+x}^{1+x} \hat{C}_{\mathbf{a}}^*(y) e^{i\pi(k+\frac{1}{2})(y-x)} dy = \frac{1}{2} \cdot \int_{-1}^1 \hat{C}_{\mathbf{a}}^*(y+x) \cdot e^{i\pi(k+\frac{1}{2})y} dy \\ &= \frac{1}{2} \cdot \int_{-1}^1 e^{-i\pi(k+\frac{1}{2})y} \cdot \hat{C}_{\mathbf{a}}^*(x-y) dy = \frac{1}{\sqrt{2}} \langle e_k | \hat{C}_{\mathbf{a}}^*(x-\cdot) \rangle_2, \end{aligned}$$

where  $\hat{C}_{\mathbf{a}}$  denotes the extension of  $C$  to a 2-antiperiodic function on  $\mathbb{R}$ . Since for every finite subset  $S \subset \mathbb{N}$ ,

$$\sum_{k \in S} |(e_k(x))^* \cdot \langle C | e_k \rangle_2|^2 \leq \sum_{k \in S} |\langle e_k | C \rangle_2|^2 \leq \sum_{k \in \mathbb{N}} |\langle e_k | C \rangle_2|^2,$$

$(|(e_k(x))^* \cdot \langle C | e_k \rangle_2|^2)_{k \in \mathbb{N}}$  is summable.

Hence, we note that

$$\begin{aligned} (C \ast_{\mathbf{a}} u)(x) &= \sum_{l \in \mathbb{N}} \langle e_{\beta(l)} | C \rangle_2 \langle e_{\beta(l)} | u \rangle_2 \cdot e_{\beta(l)}(x) = \left\langle \sum_{l \in \mathbb{N}} (e_{\beta(l)}(x))^* \cdot \langle C | e_{\beta(l)} \rangle_2 \cdot e_{\beta(l)} | u \right\rangle_2 \\ &= \left\langle \sum_{l \in \mathbb{N}} \frac{1}{\sqrt{2}} \langle e_{\beta(l)} | \hat{C}_{\mathbf{a}}^*(x-\cdot) \rangle_2 \cdot e_{\beta(l)} | u \right\rangle_2 = \frac{1}{\sqrt{2}} \left\langle \sum_{l \in \mathbb{N}} \langle e_{\beta(l)} | \hat{C}_{\mathbf{a}}^*(x-\cdot) \rangle_2 \cdot e_{\beta(l)} | u \right\rangle_2 \\ &= \frac{1}{\sqrt{2}} \langle \hat{C}_{\mathbf{a}}^*(x-\cdot) | u \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \hat{C}_{\mathbf{a}}(x-y) \cdot u(y) dy. \end{aligned}$$

where  $\beta : \mathbb{N} \rightarrow \mathbb{Z}$  is some bijection.

### 4.3 Neumann Boundary Conditions

We define the operator  $A_{0,\mathbb{N}} : D(A_{0,\mathbb{N}}) \rightarrow L^2_{\mathbb{C}}(I)$  by

$$D(A_{0,\mathbb{N}}) := \left\{ u \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \rightarrow -1} u'(x) = \lim_{x \rightarrow 1} u'(x) = 0 \right\}$$

and

$$A_{0,\mathbb{N}} u := -\frac{4}{\pi^2} u''$$

for every  $u \in D(A_{0,\mathbb{N}})$ , where  $I := (-1, 1)$ ,  $C^2(\bar{I}, \mathbb{C})$  consists of the restrictions of the elements of  $C^2(J, \mathbb{C})$  to  $I$ , where  $J$  runs through all open intervals of  $\mathbb{R}$  containing  $\bar{I}$ . Note that  $C^2(\bar{I}, \mathbb{C})$  is a *dense subspace* of  $X$ .  $A_{0,\mathbb{N}}$  is densely-defined, linear and positive symmetric.



### 4.3.1 Associated Hilbert Basis and Properties

We note that  $A_{0,\mathbb{N}}$  is a special case of a regular Sturm-Liouville operator. In particular,  $A_{0,\mathbb{N}}$  is essentially self-adjoint. The closure  $A_{\mathbb{N}}$  of  $A_{0,\mathbb{N}}$  is given by

$$A_{\mathbb{N}}u = -\frac{4}{\pi^2} u'',$$

where  $u \in W^2(I, \mathbb{C})$  where  $'$  denotes the weak derivative and  $u \in W_0^1(I, \mathbb{C})$ .  $A_{\mathbb{N}}$  has a purely discrete spectrum  $\sigma(A_{\mathbb{N}})$  consisting of simple eigenvalues,

$$\sigma(A_{\mathbb{N}}) = \{k^2 : k \in \mathbb{N}\}.$$

For every  $k \in \mathbb{N}$ , a normalized eigenvector corresponding to the eigenvalue  $k^2$  is given by

$$e_k(x) := \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 0 \\ \cos\left(\frac{k\pi}{2}(x+1)\right) & \text{if } k \neq 0 \end{cases},$$

for every  $x \in I$ ,

$$A_{0,\mathbb{N}}e_k = \frac{4}{\pi^2} \frac{\pi^2 k^2}{4} e_k = k^2 e_k.$$

Hence  $e_0, e_1, \dots$  is a Hilbert basis of  $L_{\mathbb{C}}^2(I)$ . Furthermore, we note for  $k \in \mathbb{N}^*$  and  $x \in I$  that

$$\begin{aligned} e_{2k}(x) &= (-1)^k \cos(k\pi x), \\ e_{2k-1}(x) &= (-1)^k \sin\left(\pi\left(k - \frac{1}{2}\right)x\right), \end{aligned}$$

and hence that

$$\begin{aligned} e_k &\text{ is even and periodic with period 2, for even } k \in \mathbb{N}, \\ e_k &\text{ is odd and antiperiodic with period 2, for odd } k \in \mathbb{N}. \end{aligned}$$

Also,  $\cos\left(\frac{k\pi}{2}(\text{id}_{\mathbb{R}} + 1)\right)$  is periodic with period 4 for every  $k \in \mathbb{N}$ .

### 4.3.2 Compactness of $f(A_{\mathbb{N}})$

For every  $f \in B(\sigma(A_{\mathbb{N}}), \mathbb{C})$ , if

$$(|f(k^2)|^2)_{k \in \mathbb{N}}$$

is summable, then  $f(A_{\mathbb{N}})$  is a Hilbert-Schmidt operator and hence compact. The latter is the case if

$$|f(\lambda)| \leq c \lambda^{-\alpha}$$

for every  $\lambda \in \sigma(A_{\mathbb{N}}) \setminus \{0\}$ , where  $\alpha > 1/2$ ,  $c \geq 0$ .

### 4.3.3 Properties of Simple Convolutions and Integral Representations

In the following, we give connections to the convolutions  $*_{\mathbf{p}}$  from Section 4.1, for periodic BCs, and  $*_{\mathbf{a}}$  from Section 4.2, for antiperiodic BCs. For every  $k \in \mathbb{Z}$ ,  $x \in I$ , the corresponding eigenfunctions are as follows:

$$e_k^{\mathbf{p}}(x) := \frac{1}{\sqrt{2}} e^{i\pi k x}, \quad e_k^{\mathbf{a}}(x) := \frac{1}{\sqrt{2}} e^{i\pi(k+\frac{1}{2})x}.$$

We note for even  $C \in L^2(I)$ , even  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$ ,  $k \in \mathbb{N}^*$  that

$$\langle e_k^{\mathbf{p}} | C \rangle_2 = \langle e_{-k}^{\mathbf{p}} | C \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \cos(\pi k y) C(y) dy \quad \left( \leq \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy \right),$$

$$\langle e_k^{\mathbf{p}} | u \rangle_2 = \langle e_{-k}^{\mathbf{p}} | u \rangle_2 = \frac{(-1)^k}{\sqrt{2}} \langle e_{2k} | u \rangle_2.$$

As a consequence, for  $k \in \mathbb{N}^*$

$$\langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_k^{\mathbf{p}} | u \rangle_2 e_k^{\mathbf{p}}(x) + \langle e_{-k}^{\mathbf{p}} | C \rangle_2 \langle e_{-k}^{\mathbf{p}} | u \rangle_2 e_{-k}^{\mathbf{p}}(x)$$

$$= \sqrt{2}(-1)^k \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_k^{\mathbf{p}} | u \rangle_2 e_{2k}(x) = \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_{2k} | u \rangle_2 e_{2k}(x) .$$

Hence

$$\begin{aligned} C *_{\mathbf{p}} u &= \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_k^{\mathbf{p}} | u \rangle_2 e_k^{\mathbf{p}} = \sum_{k=0}^{\infty} \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_{2k} | u \rangle_2 e_{2k} \\ &= \sum_{k=0}^{\infty} \varphi_1(k^2) \langle e_k | u \rangle_2 e_k , \end{aligned}$$

where  $\varphi_1 \in B(\sigma(A_{\mathbb{N}}), \mathbb{C})$  is defined by

$$\varphi_1(k^2) := \begin{cases} 0 & \text{if } k \in \mathbb{N} \text{ is odd} \\ \langle e_{k/2}^{\mathbf{p}} | C \rangle_2 & \text{if } k \in \mathbb{N} \text{ is even} \end{cases} ,$$

and

$$C *_{\mathbf{p}} P_{\text{even}} = \varphi_1(A_{\mathbb{N}}) ,$$

where the orthogonal projection  $P_{\text{even}} : L_{\mathbb{C}}^2(I) \rightarrow L_{\mathbb{C}}^2(I)$  is defined by

$$P_{\text{even}} h := \frac{1}{2} (h + h \circ (-\text{id}_I)) ,$$

for every  $h \in L_{\mathbb{C}}^2(I)$ .

Also, we note for even  $C \in L^2(I)$  and odd  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$ ,  $k \in \mathbb{N}$  that

$$\begin{aligned} \langle e_k^{\mathbf{a}} | C \rangle_2 &= \langle e_{-k-1}^{\mathbf{a}} | C \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \cos \left[ \pi \left( k + \frac{1}{2} \right) y \right] \cdot C(y) dy \quad \left( \leq \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy \right) , \\ \langle e_k^{\mathbf{a}} | u \rangle_2 &= - \langle e_{-k-1}^{\mathbf{a}} | u \rangle_2 = \frac{(-1)^k i}{\sqrt{2}} \langle e_{2k+1} | u \rangle_2 . \end{aligned}$$

As a consequence,

$$\langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_k^{\mathbf{a}} | u \rangle_2 e_k^{\mathbf{a}}(x) + \langle e_{-k-1}^{\mathbf{a}} | C \rangle_2 \langle e_{-k-1}^{\mathbf{a}} | u \rangle_2 e_{-k-1}^{\mathbf{a}}(x) = \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_{2k+1} | u \rangle_2 e_{2k+1}(x) .$$

Hence,

$$\begin{aligned} C *_{\mathbf{a}} u &= \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_k^{\mathbf{a}} | u \rangle_2 e_k^{\mathbf{a}} = \sum_{k=0}^{\infty} \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_{2k+1} | u \rangle_2 e_{2k+1} \\ &= \sum_{k=0}^{\infty} \varphi_2(k^2) \langle e_k | u \rangle_2 e_k , \end{aligned}$$

where  $\varphi_2 \in B(\sigma(A_{\mathbb{N}}), \mathbb{C})$  is defined by

$$\varphi_2(k^2) := \begin{cases} \langle e_{(k-1)/2}^{\mathbf{a}} | C \rangle_2 & \text{if } k \in \mathbb{N}^* \text{ is odd} \\ 0 & \text{if } k \in \mathbb{N}^* \text{ is even} \end{cases} ,$$

and

$$C *_{\mathbf{a}} P_{\text{odd}} = \varphi_2(A_{\mathbb{N}}) ,$$

where the orthogonal projection  $P_{\text{odd}} : L_{\mathbb{C}}^2(I) \rightarrow L_{\mathbb{C}}^2(I)$  is defined by

$$P_{\text{odd}} h := \frac{1}{2} (h - h \circ (-\text{id}_I)) ,$$

for every  $h \in L_{\mathbb{C}}^2(I)$ .

#### 4.3.4 Properties of Canonical Convolutions and Integral Representations

In the following,  $*_{\mathbb{N}}$  denotes the convolution in  $L^2_{\mathbb{C}}(I)$  that, according to Theorem 1, is associated to the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$ . In particular, for  $C \in L^2(I)$  and  $c \in \mathbb{R}$ ,

$$\langle e_k | C \rangle_2 = \int_{-1}^1 \cos\left(\frac{k\pi}{2}(y+1)\right) C(y) dy ,$$

is real-valued for every  $k \in \mathbb{N}$  and  $c - C *_{\mathbb{N}} \cdot$  is a bounded self-adjoint function of  $A_{\mathbb{N}}$ . Furthermore, since

$$\int_{-1}^1 C(y) dy - \langle e_k | C \rangle_2 = \int_{-1}^1 \left[ 1 - \cos\left(\frac{k\pi}{2}(y+1)\right) \right] C(y) dy ,$$

for every  $k \in \mathbb{N}^*$ , if  $C \geq 0$  and

$$c = \int_{-1}^1 C(y) dy ,$$

then the operator  $c - C *_{\mathbb{N}} \cdot$  is in particular positive.

If  $C$  is even,

$$\langle e_k | C \rangle_2 = 0 ,$$

for every odd  $k \in \mathbb{N}$ .

In addition, since for every  $C, g \in L^2_{\mathbb{C}}(I)$ ,  $x \in I$  and every finite subset  $S \subset \mathbb{N}$

$$\sum_{k \in S} |(e_k(x))^* \cdot \langle C | e_k \rangle_2|^2 \leq \sum_{k \in S} |\langle e_k | C \rangle_2|^2 \leq \sum_{k \in \mathbb{N}^*} |\langle e_k | C \rangle_2|^2 ,$$

we note that

$$(C *_{\mathbb{N}} u)(x) = \sum_{k \in \mathbb{N}} \langle e_k | C \rangle_2 \langle e_k | u \rangle_2 \cdot e_k(x) = \left\langle \sum_{k \in \mathbb{N}} (e_k(x))^* \cdot \langle C | e_k \rangle_2 \cdot e_k | u \right\rangle_2$$

and, since for  $k \in \mathbb{N}$ ,  $x, u \in \mathbb{R}$

$$\begin{aligned} & \cos\left(\frac{k\pi}{2}(x+1)\right) \cos\left(\frac{k\pi}{2}(y+1)\right) \\ &= \frac{1}{2} \left\{ \left[ \cos\left(\frac{k\pi}{2}(x-y+1)\right) + \cos\left(\frac{k\pi}{2}(x+y+1)\right) \right] \cos\left(\frac{k\pi}{2}\right) \right. \\ & \quad \left. + \left[ \sin\left(\frac{k\pi}{2}(x-y+1)\right) - \sin\left(\frac{k\pi}{2}(x+y+1)\right) \right] \sin\left(\frac{k\pi}{2}\right) \right\} , \end{aligned}$$

for even  $C$ ,  $k \in \mathbb{N}^*$ ,  $x \in I$  that

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= \cos\left(\frac{k\pi}{2}(x+1)\right) \int_{-1}^1 C^*(y) \cos\left(\frac{k\pi}{2}(y+1)\right) dy \\ &= \frac{1}{2} \cos\left(\frac{k\pi}{2}\right) \left[ \int_{-1}^1 C^*(y) \cos\left(\frac{k\pi}{2}(x-y+1)\right) dy \right. \\ & \quad \left. + \int_{-1}^1 C^*(y) \cos\left(\frac{k\pi}{2}(x+y+1)\right) dy \right] \\ &= \cos\left(\frac{k\pi}{2}\right) \int_{-1}^1 \hat{C}_{\mathbf{p}}^*(y) \cos\left(\frac{k\pi}{2}(x-y+1)\right) dy \\ &= \cos\left(\frac{k\pi}{2}\right) \int_{x-1}^{x+1} \hat{C}_{\mathbf{p}}^*(y) \cos\left(\frac{k\pi}{2}(x-y+1)\right) dy \\ &= \cos\left(\frac{k\pi}{2}\right) \int_{-1}^1 \hat{C}_{\mathbf{p}}^*(x-y) \cos\left(\frac{k\pi}{2}(y+1)\right) dy \\ &= \cos\left(\frac{k\pi}{2}\right) \langle e_k | \hat{C}_{\mathbf{p}}^*(x - \text{id}_{\mathbb{R}}) \rangle_2 , \end{aligned}$$

where  $\hat{C}_{\mathbf{p}}$  denotes the extension of  $C$  to a 2-*periodic* function on  $\mathbb{R}$ , i.e., such that

$$\hat{C}_{\mathbf{p}}(x+2) = \hat{C}_{\mathbf{p}}(x)$$

for every  $x \in I$ . Here, it has been used that  $\cos\left(\frac{k\pi}{2}(\text{id}_{\mathbb{R}} + 1)\right)$  is 2-*periodic* for *even*  $k \in \mathbb{N}^*$ .

We note for  $k \in \mathbb{N}^*$  that

$$\cos\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even and } k/2 \text{ is even,} \\ -1 & \text{if } k \text{ is even and } k/2 \text{ is odd.} \end{cases}$$

In the next step, we decompose  $C$  into  $C_1, C_2 \in L^2_{\mathbb{C}}(I)$ , where

$$C_1(x) := \frac{1}{2} [C(|x|) + C(1 - |x|)] \quad , \quad C_2(x) := \frac{1}{2} [C(|x|) - C(1 - |x|)] \quad ,$$

such that

$$C = C_1 + C_2 \quad .$$

Note that  $C_1, C_2$  are even and have a so called “half-wave symmetry,” i.e, that for every  $x \in [0, 1/2]$ :

$$\begin{aligned} C_1(1-x) &= \frac{1}{2} [C(|1-x|) + C(1 - |1-x|)] = \frac{1}{2} [C(1-x) + C(x)] = C_1(x) \quad , \\ C_2(1-x) &= \frac{1}{2} [C(|1-x|) - C(1 - |1-x|)] = \frac{1}{2} [C(1-x) - C(x)] = -C_2(x) \quad . \end{aligned}$$

As a consequence, for even  $k \in \mathbb{N}^*$ ,  $j \in \{1, 2\}$ ,

$$\begin{aligned} \langle e_k | C_j \rangle_2 &= \int_{-1}^1 \cos\left(\frac{k\pi}{2}(y+1)\right) C_j(y) dy \\ &= \int_{-1}^1 \left[ \cos\left(\frac{k\pi}{2}y\right) \cos\left(\frac{k\pi}{2}\right) - \sin\left(\frac{k\pi}{2}y\right) \sin\left(\frac{k\pi}{2}\right) \right] C_j(y) dy \\ &= \cos\left(\frac{k\pi}{2}\right) \int_{-1}^1 \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy = 2 \cos\left(\frac{k\pi}{2}\right) \int_0^1 \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy \\ &= 2 \cos\left(\frac{k\pi}{2}\right) \left[ \int_0^{1/2} \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy + \int_{1/2}^1 \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy \right] \\ &= 2 \cos\left(\frac{k\pi}{2}\right) \left[ \int_0^{1/2} \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy + \int_0^{1/2} \cos\left(\frac{k\pi}{2}(1-y)\right) C_j(1-y) dy \right] \\ &= 2 \cos\left(\frac{k\pi}{2}\right) \left[ \int_0^{1/2} \cos\left(\frac{k\pi}{2}y\right) C_j(y) dy + (-1)^{j+1} (-1)^{k/2} \int_0^{1/2} \cos(y) C_j(y) dy \right] \\ &= 2 \cos\left(\frac{k\pi}{2}\right) \left[ 1 + (-1)^{j+1} (-1)^{k/2} \right] \int_0^{1/2} \cos(y) C_j(y) dy \quad , \end{aligned}$$

and hence

$$\langle e_k | C_j \rangle_2 = \begin{cases} 0 & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even, } k/2 \text{ odd and } j = 1 \\ 0 & \text{if } k \text{ even, } k/2 \text{ is even and } j = 2 \end{cases} \quad .$$

From this, we conclude for even  $k \in \mathbb{N}^*$ ,  $x \in I$  that

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= (e_k(x))^* \cdot \langle C_1 | e_k \rangle_2 + (e_k(x))^* \cdot \langle C_2 | e_k \rangle_2 \\ &= \langle e_k | \hat{C}_{1,\mathbf{p}}^*(x - \text{id}_{\mathbb{R}}) \rangle_2 - \langle e_k | \hat{C}_{2,\mathbf{p}}^*(x - \text{id}_{\mathbb{R}}) \rangle_2 \quad , \end{aligned} \tag{4.1}$$

where for every  $j \in \{1, 2\}$ ,  $\hat{C}_{j,\mathbf{p}}$  denotes the extension of  $C_j$  to a 2-*periodic* function on  $\mathbb{R}$ .

In the following, we extend (4.1) to *odd*  $k \in \mathbb{N}^*$ . For this purpose, we note for every even  $g \in L_{\mathbb{C}}^2(I)$  that its 2-periodic extension  $\hat{g}$  is even, too. For the proof, let  $l \in \mathbb{N}$ ,  $x \in [-1 - 2l, -2l]$ . Then

$$\hat{u}(x) = u(x + 2l) = u(-x - 2l) = \hat{u}(-x - 2l + 2l) = \hat{u}(-x) .$$

Hence it follows for even  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$  and odd  $k \in \mathbb{N}^*$  that

$$\langle e_k | \hat{u}(x - \cdot) \rangle_2 = - \langle e_k | \hat{u}(-x - \cdot) \rangle_2 ,$$

which implies that

$$\langle e_k | \hat{u}(x - \cdot) + \hat{u}(-x - \cdot) \rangle_2 = 0 .$$

On the other hand, for even  $k \in \mathbb{N}^*$

$$\langle e_k | \hat{u}(x - \cdot) \rangle_2 = \langle e_k | \hat{u}(-x - \cdot) \rangle_2 .$$

As a consequence,

$$\langle e_k | \frac{1}{2} [\hat{u}(x - \cdot) + \hat{u}(-x - \cdot)] \rangle_2 = \begin{cases} 0 & \text{if } k \in \mathbb{N}^* \text{ is odd} \\ \langle e_k | \hat{u}(x - \cdot) \rangle_2 & \text{if } k \in \mathbb{N}^* \text{ is even} \end{cases} .$$

Therefore, we conclude from (4.1) that for *even*  $C$  and  $k \in \mathbb{N}^*$ ,  $x \in I$

$$(e_k(x))^* \cdot \langle C | e_k \rangle_2 = \langle e_k | \frac{1}{2} [\hat{C}_{1,p}^*(x - \cdot) + \hat{C}_{1,p}(-x - \cdot) - \hat{C}_{2,p}^*(x - \cdot) - \hat{C}_{2,p}^*(-x - \cdot)] \rangle_2 .$$

Furthermore,

$$\begin{aligned} & \langle e_0 | \frac{1}{2} [\hat{C}_{1,p}^*(x - \cdot) + \hat{C}_{1,p}(-x - \cdot) - \hat{C}_{2,p}^*(x - \cdot) - \hat{C}_{2,p}^*(-x - \cdot)] \rangle_2 \\ &= 2^{-3/2} \int_{-1}^1 \hat{C}_{1,p}^*(x - y) dy + 2^{-3/2} \int_{-1}^1 \hat{C}_{1,p}(-x - y) dy \\ & \quad - 2^{-3/2} \int_{-1}^1 \hat{C}_{2,p}^*(x - y) dy - 2^{-3/2} \int_{-1}^1 \hat{C}_{2,p}^*(-x - y) dy \\ &= 2^{-3/2} \int_{-1}^1 \hat{C}_{1,p}^*(y - x) dy + 2^{-3/2} \int_{-1}^1 \hat{C}_{1,p}(y + x) dy \\ & \quad - 2^{-3/2} \int_{-1}^1 \hat{C}_{2,p}^*(y - x) dy - 2^{-3/2} \int_{-1}^1 \hat{C}_{2,p}^*(y + x) dy \\ &= 2^{-3/2} \int_{-1}^1 C_1^*(y) dy + 2^{-3/2} \int_{-1}^1 C_1^*(y) dy \\ & \quad - 2^{-3/2} \int_{-1}^1 C_2^*(y) dy - 2^{-3/2} \int_{-1}^1 C_2^*(y) dy \\ &= 2^{-1/2} \int_{-1}^1 C_1^*(y) dy - 2^{-1/2} \int_{-1}^1 C_2^*(y) dy \\ &= (e_0(x))^* \cdot \langle f | e_0 \rangle_2 + \frac{\sqrt{2}-1}{2} \int_{-1}^1 C_1^*(y) dy - \frac{\sqrt{2}+1}{2} \int_{-1}^1 C_2^*(y) dy \end{aligned}$$

and hence

$$(e_0(x))^* \cdot \langle C | e_0 \rangle_2 e_0 = \langle e_0 | \frac{1}{2} [\hat{C}_{1,p}^*(x - \cdot) + \hat{C}_{1,p}(-x - \cdot) - \hat{C}_{2,p}^*(x - \cdot) - \hat{C}_{2,p}^*(-x - \cdot)] \rangle_2 e_0 + k_{N,C} e_0 ,$$

where

$$k_{N,C} := -\frac{\sqrt{2}-1}{2} \int_{-1}^1 C_1^*(y) dy + \frac{\sqrt{2}+1}{2} \int_{-1}^1 C_2^*(y) dy .$$

As a consequence,

$$C *_{\mathbb{N}} u(x) = \sum_{k \in \mathbb{N}} \langle e_k | C \rangle_2 \langle e_k | u \rangle_2 \cdot e_k(x) = \left\langle \sum_{k \in \mathbb{N}} (e_k(x))^* \cdot \langle C | e_k \rangle_2 \cdot e_k | u \right\rangle_2$$

$$\begin{aligned}
&= \left\langle \sum_{k \in \mathbb{N}} \langle e_k | \frac{1}{2} [C_1^*(x - \cdot) + C_1^*(-x - \cdot) - C_2^*(x - \cdot) - C_2^*(-x - \cdot)] \rangle_2 \cdot e_k | g \right\rangle_2 + k_{N,C} \langle e_0 | u \rangle_2 \\
&= \frac{1}{2} \int_{-1}^1 C_1(x - y) u(y) dy + \frac{1}{2} \int_{-1}^1 C_1(-x - y) u(y) dy \\
&\quad - \frac{1}{2} \int_{-1}^1 C_2(x - y) u(y) dy - \frac{1}{2} \int_{-1}^1 C_2(-x - y) u(y) dy + k_{N,C} \langle e_0 | u \rangle_2 .
\end{aligned}$$

#### 4.3.5 Connections to the Standard Fourier Expansion

In the following, we connect the expansion with respect to the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$  to that expansion with respect to the Hilbert basis  $(e_k^{\mathbf{p}})_{k \in \mathbb{Z}}$  of  $L_{\mathbb{C}}^2((-2, 2))$ , where

$$e_k^{\mathbf{p}}(x) := \frac{1}{2} e^{i\pi k x / 2} ,$$

for every  $x \in (-2, 2)$ . We note for  $k \in \mathbb{N}^*$ ,  $x \in I$  that

$$e_k(x) = e^{\frac{ik\pi}{2}} \cdot e_k^{\mathbf{p}}(x) + e^{-\frac{ik\pi}{2}} \cdot e_{-k}^{\mathbf{p}}(x) .$$

This implies for  $u \in L_{\mathbb{C}}^2(I)$  that

$$\langle e_k | u \rangle_2 = e^{-\frac{ik\pi}{2}} \langle e_k^{\mathbf{p}} | \bar{u} \rangle_{\bar{I},2} + e^{\frac{ik\pi}{2}} \langle e_{-k}^{\mathbf{p}} | \bar{u} \rangle_{\bar{I},2}$$

where  $\bar{I} := (-2, 2)$ ,  $\langle | \rangle_{\bar{I},2}$  denotes the scalar product on  $L_{\mathbb{C}}^2(\bar{I})$  and  $\bar{u} \in L_{\mathbb{C}}^2((-2, 2))$  is defined by

$$\bar{u}(x) := \begin{cases} 0 & \text{if } x \in (-2, -1) \\ u(x) & \text{if } x \in (-1, 1) \\ 0 & \text{if } x \in (1, 2) \end{cases} ,$$

for a.e.  $x \in (-2, 2)$ . Furthermore, for  $u \in L_{\mathbb{C}}^2(I)$  and on  $I$

$$u = \langle e_0 | u \rangle_2 e_0 + \sum_{k=1}^{\infty} \langle e_k | u \rangle_2 e_k = \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{p}} | u \rangle_2 \cdot e_k^{\mathbf{p}} + \sum_{k \in \mathbb{Z}} e^{ik\pi} \langle e_{-k}^{\mathbf{p}} | u \rangle_2 \cdot e_k^{\mathbf{p}} .$$

We note for  $k \in \mathbb{Z}$  that

$$e^{ik\pi} \langle e_{-k}^{\mathbf{p}} | u \rangle_2 = \int_{-2}^{-1} \frac{1}{2} e^{-i\pi k y / 2} u(-y - 2) dy + \int_1^2 \frac{1}{2} e^{-i\pi k y / 2} u(-y + 2) dy .$$

Hence on  $I$

$$u = \sum_{k=0}^{\infty} \langle e_k | u \rangle_2 e_k = \sum_{k \in \mathbb{Z}} \left[ \int_{-2}^2 (e_k^{\mathbf{p}}(y))^* \cdot u_e(y) dy \right] \cdot e_k^{\mathbf{p}} ,$$

where

$$u_e(x) := \begin{cases} u(-x - 2) & \text{if } x \in (-2, -1) \\ u(x) & \text{if } x \in (-1, 1) \\ u(-x + 2) & \text{if } x \in (1, 2) \end{cases} .$$

for a.e.  $x \in (-2, 2)$ .

#### 4.4 Dirichlet Boundary Conditions

We define the operator  $A_{0,D} : D(A_{0,D}) \rightarrow L_{\mathbb{C}}^2(I)$  by

$$D(A_{0,D}) := \left\{ u \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \rightarrow -1} u(x) = \lim_{x \rightarrow 1} u(x) = 0 \right\}$$

and

$$A_{0,D} u := -\frac{4}{\pi^2} u''$$

for every  $u \in D(A_{0,D})$ , where  $I := (-1, 1)$ ,  $C^2(\bar{I}, \mathbb{C})$  consists of the restrictions of the elements of  $C^2(J, \mathbb{C})$  to  $I$ , where  $J$  runs through all open intervals of  $\mathbb{R}$  containing  $\bar{I}$ . Note that  $C^2(\bar{I}, \mathbb{C})$  is a *dense subspace* of  $X$ .  $A_{0,D}$  is densely-defined, linear and positive symmetric.

#### 4.4.1 Associated Hilbert Basis and Properties

We note that  $A_{0,D}$  is a special case of a regular Sturm-Liouville operator. In particular,  $A_{0,D}$  is essentially self-adjoint. The closure  $A_D$  of  $A_{0,D}$ , given by

$$A_D u = -\frac{4}{\pi^2} u'',$$

where  $u \in W_0^2(I, \mathbb{C})$  and  $'$  denotes the weak derivative.  $A_D$  has a purely discrete spectrum  $\sigma(A_D)$  consisting of simple eigenvalues,

$$\sigma(A_D) = \{k^2 : k \in \mathbb{N}^*\}.$$

For every  $k \in \mathbb{N}^*$ , a normalized eigenvector corresponding to the eigenvalue  $k^2$  is given by

$$e_k(x) := \sin\left(\frac{k\pi}{2}(x+1)\right),$$

for every  $x \in I$ ,

$$A_{0,D} e_k = \frac{4}{\pi^2} \frac{\pi^2 k^2}{4} e_k = k^2 e_k,$$

Hence  $e_1, e_2, \dots$  is a Hilbert basis of  $L_{\mathbb{C}}^2(I)$ . Furthermore, we note for  $k \in \mathbb{N}^*$ ,  $l \in \mathbb{N}$  and  $x \in I$  that

$$\begin{aligned} e_{2k}(x) &= (-1)^k \sin(k\pi x), \\ e_{2l+1}(x) &= (-1)^l \cos\left(\pi\left(l + \frac{1}{2}\right)x\right) \end{aligned}$$

and hence that

$$\begin{aligned} e_k &\text{ is odd and periodic with period 2, for even } k \in \mathbb{N}^*, \\ e_k &\text{ is even and antiperiodic with period 2, for odd } k \in \mathbb{N}^*. \end{aligned}$$

Also,  $\sin\left(\frac{k\pi}{2}(\text{id}_{\mathbb{R}} + 1)\right)$  is periodic with period 4 for every  $k \in \mathbb{N}^*$ .

#### 4.4.2 Compactness of $f(A_D)$

For every  $f \in B(\sigma(A_D), \mathbb{C})$ , if

$$(|f(k^2)|^2)_{k \in \mathbb{N}^*}$$

is summable, then  $f(A_D)$  is a Hilbert-Schmidt operator and hence compact. The latter is the case if

$$|f(\lambda)| \leq c \lambda^{-\alpha}$$

for every  $\lambda \in \sigma(A_D)$ , where  $\alpha > 1/2$ ,  $c \geq 0$ .

#### 4.4.3 Properties of Simple Convolutions and Integral Representations

In the following, we give connections to the convolutions  $*_p$  from Section 4.1, for periodic BCs, and  $*_a$  from Section 4.2, for antiperiodic BCs. For every  $k \in \mathbb{Z}$ ,  $x \in I$ , the corresponding eigenfunctions are as follows:

$$e_k^p(x) := \frac{1}{\sqrt{2}} e^{i\pi k x}, \quad e_k^a(x) := \frac{1}{\sqrt{2}} e^{i\pi(k+\frac{1}{2})x}.$$

We note for even  $C \in L^2(I)$ , odd  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$ ,  $k \in \mathbb{N}^*$  that

$$\begin{aligned} \langle e_k^p | C \rangle_2 &= \langle e_{-k}^p | C \rangle_2 = \frac{1}{\sqrt{2}} \int_{-1}^1 \cos(\pi k y) C(y) dy \quad \left( \leq \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy \right), \\ \langle e_k^p | u \rangle_2 &= -\langle e_{-k}^p | u \rangle_2 = \frac{(-1)^{k+1} i}{\sqrt{2}} \langle e_{2k} | u \rangle_2. \end{aligned}$$

As a consequence, for  $k \in \mathbb{N}^*$

$$\begin{aligned} &\langle e_k^p | C \rangle_2 \langle e_k^p | u \rangle_2 e_k^p(x) + \langle e_{-k}^p | C \rangle_2 \langle e_{-k}^p | u \rangle_2 e_{-k}^p(x) \\ &= i\sqrt{2}(-1)^k \langle e_k^p | C \rangle_2 \langle e_k^p | u \rangle_2 e_{2k}(x) = \langle e_k^p | C \rangle_2 \langle e_{2k} | u \rangle_2 e_{2k}(x). \end{aligned}$$

Hence

$$\begin{aligned} C *_{\mathbf{p}} u &= \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_k^{\mathbf{p}} | u \rangle_2 e_k^{\mathbf{p}} = \sum_{k=1}^{\infty} \langle e_k^{\mathbf{p}} | C \rangle_2 \langle e_{2k} | u \rangle_2 e_{2k} \\ &= \sum_{k=1}^{\infty} \varphi_1(k^2) \langle e_k | u \rangle_2 e_k , \end{aligned}$$

where  $\varphi_1 \in B(\sigma(A_{\mathbf{D}}), \mathbb{C})$  is defined by

$$\varphi_1(k^2) := \begin{cases} 0 & \text{if } k \in \mathbb{N}^* \text{ is odd} \\ \langle e_{k/2}^{\mathbf{p}} | C \rangle_2 & \text{if } k \in \mathbb{N}^* \text{ is even} \end{cases} ,$$

and

$$C *_{\mathbf{p}} P_{\text{odd}} = \varphi_1(A_{\mathbf{D}}) .$$

Also, we note for even  $C \in L^2(I)$  and even  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$ ,  $k \in \mathbb{N}$  that

$$\begin{aligned} \langle e_k^{\mathbf{a}} | C \rangle_2 &= \langle e_{-k-1}^{\mathbf{a}} | C \rangle_2 = \frac{(-1)^k}{\sqrt{2}} \langle e_{2k+1} | C \rangle_2 \quad \left( \leq \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy \right) , \\ \langle e_k^{\mathbf{a}} | u \rangle_2 &= \langle e_{-k-1}^{\mathbf{a}} | u \rangle_2 = \frac{(-1)^k}{\sqrt{2}} \langle e_{2k+1} | u \rangle_2 . \end{aligned}$$

As a consequence,

$$\begin{aligned} &\langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_k^{\mathbf{a}} | u \rangle_2 e_k^{\mathbf{a}}(x) + \langle e_{-k-1}^{\mathbf{a}} | C \rangle_2 \langle e_{-k-1}^{\mathbf{a}} | u \rangle_2 e_{-k-1}^{\mathbf{a}}(x) \\ &= \sqrt{2} (-1)^k \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_k^{\mathbf{a}} | u \rangle_2 e_{2k+1}(x) = \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_{2k+1} | u \rangle_2 e_{2k+1}(x) . \end{aligned}$$

Hence

$$\begin{aligned} C *_{\mathbf{a}} u &= \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_k^{\mathbf{a}} | u \rangle_2 e_k^{\mathbf{a}} = \sum_{k=0}^{\infty} \langle e_k^{\mathbf{a}} | C \rangle_2 \langle e_{2k+1} | u \rangle_2 e_{2k+1} \\ &= \sum_{k=1}^{\infty} \varphi_2(k^2) \langle e_k | u \rangle_2 e_k , \end{aligned}$$

where  $\varphi_2 \in B(\sigma(A_{\mathbf{D}}), \mathbb{C})$  is defined by

$$\varphi_2(k^2) := \begin{cases} \langle e_{(k-1)/2}^{\mathbf{a}} | C \rangle_2 & \text{if } k \in \mathbb{N}^* \text{ is odd} \\ 0 & \text{if } k \in \mathbb{N}^* \text{ is even} \end{cases} ,$$

and

$$C *_{\mathbf{a}} P_{\text{even}} = \varphi_2(A_{\mathbf{D}}) .$$

#### 4.4.4 Properties of Canonical Convolutions and Integral Representations

In the following,  $*_{\mathbf{D}}$  denotes the convolution in  $L_{\mathbb{C}}^2(I)$  that, according to Theorem 1, is associated to the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$ . In particular, for  $C \in L^2(I)$  and  $c \in \mathbb{R}$ ,

$$\langle e_k | C \rangle_2 = \int_{-1}^1 \sin\left(\frac{k\pi}{2}(y+1)\right) C(y) dy ,$$

is real-valued for every  $k \in \mathbb{N}^*$  and  $c - C *_{\mathbf{D}} \cdot$  is a bounded self-adjoint function of  $A_{\mathbf{D}}$ . Furthermore, since

$$\int_{-1}^1 C dy - \langle e_k | C \rangle_2 = \int_{-1}^1 \left[ 1 - \sin\left(\frac{k\pi}{2}(y+1)\right) \right] C(y) dy ,$$

for every  $k \in \mathbb{N}^*$ , if  $C \geq 0$  and

$$c = \int_{-1}^1 C(y) dy ,$$



then the operator  $c - C *_D \cdot$  is in particular positive.

If  $C$  is even,

$$\langle e_k | C \rangle_2 = 0 ,$$

for every even  $k \in \mathbb{N}^*$ .

In addition, since for every  $C, g \in L^2_{\mathbb{C}}(I)$ ,  $x \in I$  and every finite subset  $S \subset \mathbb{N}^*$

$$\sum_{k \in S} |(e_k(x))^* \cdot \langle C | e_k \rangle_2|^2 \leq \sum_{k \in S} |\langle e_k | C \rangle_2|^2 \leq \sum_{k \in \mathbb{N}^*} |\langle e_k | C \rangle_2|^2 ,$$

we note that

$$(C *_D u)(x) = \sum_{k \in \mathbb{N}^*} \langle e_k | C \rangle_2 \langle e_k | u \rangle_2 \cdot e_k(x) = \left\langle \sum_{k \in \mathbb{N}^*} (e_k(x))^* \cdot \langle C | e_k \rangle_2 \cdot e_k | u \right\rangle_2$$

and, since for  $k \in \mathbb{N}^*$ ,  $x, u \in \mathbb{R}$

$$\begin{aligned} & \sin\left(\frac{k\pi}{2}(x+1)\right) \sin\left(\frac{k\pi}{2}(y+1)\right) \\ &= \frac{1}{2} \left\{ \left[ \cos\left(\frac{k\pi}{2}(x-y+1)\right) - \cos\left(\frac{k\pi}{2}(x+y+1)\right) \right] \cos\left(\frac{k\pi}{2}\right) \right. \\ & \quad \left. + \left[ \sin\left(\frac{k\pi}{2}(x-y+1)\right) + \sin\left(\frac{k\pi}{2}(x+y+1)\right) \right] \sin\left(\frac{k\pi}{2}\right) \right\} , \end{aligned}$$

for even  $C$ ,  $k \in \mathbb{N}^*$ ,  $x \in I$  that

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= \sin\left(\frac{k\pi}{2}(x+1)\right) \int_{-1}^1 C^*(y) \sin\left(\frac{k\pi}{2}(y+1)\right) dy \\ &= \sin\left(\frac{k\pi}{2}\right) \int_{-1}^1 \hat{C}_a^*(y) \sin\left(\frac{k\pi}{2}(x-y+1)\right) dy \\ &= \sin\left(\frac{k\pi}{2}\right) \int_{x-1}^{x+1} \hat{C}_a^*(y) \sin\left(\frac{k\pi}{2}(x-y+1)\right) dy \\ &= \sin\left(\frac{k\pi}{2}\right) \int_{-1}^1 \hat{C}_a^*(x-y) \sin\left(\frac{k\pi}{2}(y+1)\right) dy \\ &= \sin\left(\frac{k\pi}{2}\right) \langle e_k | \hat{C}_a^*(x - \text{id}_{\mathbb{R}}) \rangle_2 , \end{aligned}$$

where  $\hat{C}_a$  denotes the extension of  $C$  to a 2-antiperiodic function on  $\mathbb{R}$ , i.e., such that

$$\hat{C}_a(x+2) = -\hat{C}_a(x)$$

for every  $x \in I$ . Here, it has been used that  $\sin\left(\frac{k\pi}{2}(\text{id}_{\mathbb{R}} + 1)\right)$  is 2-antiperiodic for odd  $k \in \mathbb{N}^*$ .

We note for  $k \in \mathbb{N}^*$  that

$$\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd and } (k-1)/2 \text{ is even} \\ -1 & \text{if } k \text{ is odd and } (k-1)/2 \text{ is odd} \end{cases} .$$

As a consequence, if we denote by  $P$  the orthogonal projection onto the closure of the subspace

$$\text{Span}(\{e_{4l+1} : l \in \mathbb{N}\}) ,$$

then for even  $C$  and odd  $k \in \mathbb{N}^*$ ,  $x \in I$

$$\begin{aligned} (e_k(x))^* \cdot \langle C | e_k \rangle_2 &= (e_k(x))^* \langle PC | e_k \rangle_2 + (e_k(x))^* \langle C - PC | e_k \rangle_2 \\ &= \langle e_k | (\widehat{PC}_a)^*(x - \cdot) - (\hat{C}_a - \widehat{PC}_a)^*(x - \cdot) \rangle_2 , \end{aligned} \tag{4.2}$$

where  $\widehat{\text{PC}}_{\mathbf{a}}$  denotes the extension of PC to a 2-*antiperiodic* function on  $\mathbb{R}$ . Here, we used for every odd  $k \in \mathbb{N}^*$  that  $e_k$  is even and hence that PC and  $C - \text{PC}$  are even.

In the following, we extend (4.2) to *even*  $k \in \mathbb{N}^*$ . For this purpose, we note for every even  $u \in L_{\mathbb{C}}^2(I)$  that its 2-anti-periodic extension  $\hat{u}$  is even, too. For the proof, let  $l \in \mathbb{N}$ ,  $x \in [-1 - 2l, -2l]$ . Then

$$\hat{u}(x) = (-1)^l u(x + 2l) = (-1)^l u(-x - 2l) = (-1)^l \cdot (-1)^l \hat{u}(-x - 2l + 2l) = \hat{u}(-x) .$$

Hence it follows for even  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$  and even  $k \in \mathbb{N}^*$  that

$$\langle e_k | \hat{u}(x - \cdot) \rangle_2 = - \langle e_k | \hat{u}(-x - \cdot) \rangle_2 ,$$

which implies that

$$\langle e_k | \hat{u}(x - \cdot) + \hat{u}(-x - \cdot) \rangle_2 = 0 .$$

On the other hand, for odd  $k \in \mathbb{N}^*$

$$\langle e_k | \hat{u}(x - \cdot) \rangle_2 = \langle e_k | \hat{u}(-x - \cdot) \rangle_2 .$$

As a consequence,

$$\langle e_k | \frac{1}{2} [\hat{u}(x - \cdot) + \hat{u}(-x - \cdot)] \rangle_2 = \begin{cases} 0 & \text{if } k \in \mathbb{N}^* \text{ is even} \\ \langle e_k | \hat{u}(x - \cdot) \rangle_2 & \text{if } k \in \mathbb{N}^* \text{ is odd} \end{cases} .$$

Therefore, we conclude from (4.2) that for *even*  $C$  and  $k \in \mathbb{N}^*$ ,  $x \in I$

$$\begin{aligned} & (e_k(x))^* \cdot \langle C | e_k \rangle_2 \\ &= \langle e_k | \frac{1}{2} [(\widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) + (\widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot) - (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) - (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot)] \rangle_2 \end{aligned}$$

and hence that

$$\begin{aligned} (C *_D u)(x) &= \sum_{k \in \mathbb{N}^*} \langle e_k | C \rangle_2 \langle e_k | u \rangle_2 \cdot e_k(x) \\ &= \left\langle \sum_{k \in \mathbb{N}^*} (e_k(x))^* \cdot \langle C | e_k \rangle_2 \cdot e_k | u \right\rangle_2 \\ &= \left\langle \sum_{k \in \mathbb{N}^*} \langle e_k | \frac{1}{2} [(\widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) + (\widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot) \right. \\ &\quad \left. - (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) - (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot)] \rangle_2 \cdot e_k | g \right\rangle_2 \\ &= \frac{1}{2} \langle (\widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) | u \rangle_2 + \frac{1}{2} \langle (\widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot) | u \rangle_2 \\ &\quad - \frac{1}{2} \langle (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(x - \cdot) | u \rangle_2 - \frac{1}{2} \langle (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})^*(-x - \cdot) | u \rangle_2 \\ &= \frac{1}{2} \int_{-1}^1 (\widehat{\text{PC}}_{\mathbf{a}})(x - y) u(y) dy + \frac{1}{2} \int_{-1}^1 (\widehat{\text{PC}}_{\mathbf{a}})(-x - y) u(y) dy \\ &\quad - \frac{1}{2} \int_{-1}^1 (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})(x - y) u(y) dy - \frac{1}{2} \int_{-1}^1 (\hat{C}_{\mathbf{a}} - \widehat{\text{PC}}_{\mathbf{a}})(-x - y) u(y) dy . \end{aligned}$$

#### 4.4.5 Representation of the Projection Present in the Canonical Convolution

In the following, we give a representation of P, which is independent of the orthonormal system used in its definition.

For  $u \in L_{\mathbb{C}}^2(I)$ ,  $x \in I$ , we note that

$$\sum_{k=0}^n \langle e_{4k+1} | u \rangle e_{4k+1}(x) = \left\langle \sum_{k=0}^n e_{4k+1}(x) \cdot e_{4k+1} | u \right\rangle .$$

Furthermore, for  $y \in I$ ,

$$e_{4k+1}(x) e_{4k+1}(y) = \sin \left( \frac{(4k+1)\pi}{2} (x+1) \right) \sin \left( \frac{(4k+1)\pi}{2} (y+1) \right)$$

$$= \frac{1}{2} \left[ \sin \left( \frac{(4k+1)\pi}{2} (x-y+1) \right) + \sin \left( \frac{(4k+1)\pi}{2} (x+y+1) \right) \right] .$$

Since for  $a \in \mathbb{R}$ ,

$$\begin{aligned} \sum_{k=0}^n \sin((4k+1)a) &= \sum_{k=0}^n [\sin(4ka) \cos(a) + \cos(4ka) \sin(a)] \\ &= \cos(a) \sum_{k=0}^n \sin(4ka) + \sin(a) \sum_{k=0}^n \cos(4ka) , \end{aligned}$$

for  $n \in \mathbb{N}^*$ ,  $b \in \mathbb{C}$  satisfying  $b \neq 2\pi l, l \in \mathbb{Z}$ ,

$$\begin{aligned} \sum_{k=0}^n \sin(kb) &= \frac{1}{2i} \left[ \sum_{k=0}^n e^{ikb} - \sum_{k=0}^n e^{-ikb} \right] = \frac{1}{2i} \left[ \sum_{k=0}^n (e^{ib})^k - \sum_{k=0}^n (e^{-ib})^k \right] \\ &= \frac{1}{2 \sin(b/2)} \left\{ \sin \left( \frac{b}{2} \right) + \sin \left[ \left( n + \frac{1}{2} \right) b \right] \right\} \end{aligned}$$

and hence if  $a \neq l\pi/2, l \in \mathbb{Z}$ ,

$$\sum_{k=0}^n \sin((4k+1)a) = \frac{\sin[(2n+1)a] \sin[2(n+1)a]}{\sin(2a)} ,$$

we conclude that

$$\begin{aligned} &\sum_{k=0}^n e_{4k+1}(x) e_{4k+1}(y) \\ &= \frac{1}{2} \frac{\sin \left( \frac{(2n+1)\pi}{2} (x-y+1) \right) \sin \left( \frac{2(n+1)\pi}{2} (x-y+1) \right)}{\sin[\pi(x-y+1)]} \\ &\quad + \frac{1}{2} \frac{\sin \left( \frac{(2n+1)\pi}{2} (x+y+1) \right) \sin \left( \frac{2(n+1)\pi}{2} (x+y+1) \right)}{\sin[\pi(x+y+1)]} , \end{aligned}$$

if both

$$x-y, x+y \notin \mathbb{Z} .$$

As a consequence,

$$\begin{aligned} &\sum_{k=0}^n \langle e_{4k+1} | u \rangle e_{4k+1}(x) \\ &\quad + \frac{1}{2} \int_{-1}^1 \frac{\sin \left( \frac{(2n+1)\pi}{2} (x+y+1) \right) \sin \left( \frac{2(n+1)\pi}{2} (x+y+1) \right)}{\sin[\pi(x+y+1)]} u(y) dy \\ &= \frac{1}{2} \int_{-1}^1 \frac{\sin \left( \frac{(2n+1)\pi}{2} (x-y+1) \right) \sin \left( \frac{2(n+1)\pi}{2} (x-y+1) \right)}{\sin[\pi(x-y+1)]} u(y) dy \\ &\quad + \frac{1}{2} \int_{-1}^1 \frac{\sin \left( \frac{(2n+1)\pi}{2} (x-y+1) \right) \sin \left( \frac{2(n+1)\pi}{2} (x-y+1) \right)}{\sin[\pi(x-y+1)]} u(-y) dy , \end{aligned}$$

and hence

$$\text{Pu} = \lim_{n \rightarrow \infty} \frac{\sin \left[ \pi \left( n + \frac{1}{2} \right) (\text{id}_I + 1) \right] \sin [\pi(n+1)(\text{id}_I + 1)]}{\sin[\pi(\text{id}_I + 1)]} * \frac{1}{2} [u + u \circ (-\text{id}_I)] ,$$

where  $*$  denotes the integral convolution on  $I$ , and the limit is to be performed in  $L_{\mathbb{C}}^2(I)$ .

#### 4.4.6 Connections to the Standard Fourier Expansion

In the following, we connect the expansion with respect to the Hilbert basis  $(e_k)_{k \in \mathbb{N}^*}$  to that expansion with respect to the Hilbert basis  $(e_k^{\mathbf{p}})_{k \in \mathbb{Z}}$  of  $L_{\mathbb{C}}^2((-2, 2))$ , where

$$e_k^{\mathbf{p}}(x) := \frac{1}{2} e^{i\pi kx/2},$$

for every  $x \in (-2, 2)$ . We note for  $k \in \mathbb{N}^*$ ,  $x \in I$  that

$$e_k(x) = \frac{e^{\frac{ik\pi}{2}}}{i} \cdot e_k^{\mathbf{p}}(x) - \frac{e^{-\frac{ik\pi}{2}}}{i} \cdot e_{-k}^{\mathbf{p}}(x).$$

This implies for  $u \in L_{\mathbb{C}}^2(I)$  that

$$\langle e_k | u \rangle_2 = -\frac{e^{-\frac{ik\pi}{2}}}{i} \langle e_k^{\mathbf{p}} | \bar{u} \rangle_{\bar{I}, 2} + \frac{e^{\frac{ik\pi}{2}}}{i} \langle e_{-k}^{\mathbf{p}} | \bar{u} \rangle_{\bar{I}, 2}$$

where  $\bar{I} := (-2, 2)$ ,  $\langle | \rangle_{\bar{I}, 2}$  denotes the scalar product on  $L_{\mathbb{C}}^2(\bar{I})$  and  $\bar{u} \in L_{\mathbb{C}}^2((-2, 2))$  is defined by

$$\bar{u}(x) := \begin{cases} 0 & \text{if } x \in (-2, -1) \\ u(x) & \text{if } x \in (-1, 1) \\ 0 & \text{if } x \in (1, 2) \end{cases},$$

for a.e.  $x \in (-2, 2)$ . Furthermore, for  $u \in L_{\mathbb{C}}^2(I)$  and on  $I$

$$u = \sum_{k=1}^{\infty} \langle e_k | u \rangle_2 e_k = \sum_{k \in \mathbb{Z}} \langle e_k^{\mathbf{p}} | u \rangle_2 \cdot e_k^{\mathbf{p}} - \sum_{k \in \mathbb{Z}} e^{ik\pi} \langle e_{-k}^{\mathbf{p}} | u \rangle_2 \cdot e_k^{\mathbf{p}}.$$

We note for  $k \in \mathbb{Z}$  that

$$e^{ik\pi} \langle e_{-k}^{\mathbf{p}} | u \rangle_2 = \int_{-2}^{-1} \frac{1}{2} e^{-i\pi ky/2} u(-y-2) dy + \int_1^2 \frac{1}{2} e^{-i\pi ky/2} u(-y+2) dy.$$

Hence on  $I$

$$u = \sum_{k=1}^{\infty} \langle e_k | u \rangle_2 e_k = \sum_{k \in \mathbb{Z}} \left[ \int_{-2}^2 (e_k^{\mathbf{p}}(x))^* \cdot u_e(x) dx \right] \cdot e_k^{\mathbf{p}},$$

where

$$u_e(x) := \begin{cases} -u(-x-2) & \text{if } x \in (-2, -1) \\ u(x) & \text{if } x \in (-1, 1) \\ -u(-x+2) & \text{if } x \in (1, 2) \end{cases}.$$

for a.e.  $x \in (-2, 2)$ .

## 5 Numerical Experiments

Recalling the governing equation (2.3), we numerically solve the following nonlocal equation

$$u_{tt}(x, t) + \varphi(A_{\mathbf{BC}})u(x, t) = b(x, t), \quad (x, t) \in \Omega \times J, \quad (5.1a)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (5.1b)$$

$$u_t(x, 0) = v_0(x) \quad x \in \Omega, \quad (5.1c)$$

where  $J = (0, T)$  is some finite time interval,  $\Omega = (-1, 1)$ ,  $b$  is a given source term, and  $u_0$  and  $v_0$  are given initial conditions. The choice of the subscript  $\mathbf{BC} \in \{\mathbf{p}, \mathbf{a}, \mathbf{N}, \mathbf{D}\}$  is determined by the BCs that are to be satisfied at the boundary of the physical domain  $(-1, 1)$ . This, in turn, determines the function of the classical operator  $\varphi(A_{\mathbf{BC}})$  as described in Table 5.1 where we have defined

$$c := \frac{1}{\sqrt{2}} \int_{-1}^1 C(y) dy. \quad (5.2)$$

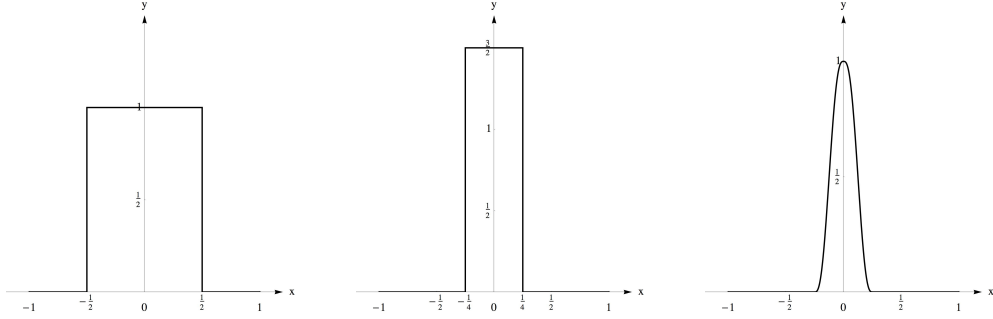


Figure 5.1: Micromodulus function  $C(x)$  (left). Discontinuous (middle) and continuous (right) initial displacement functions  $u_{0,\text{disc}}(x)$  and  $u_{0,\text{cont}}(x)$ , respectively.

BC	$\varphi(A_{\text{BC}})u(x, t)$	BCs enforced
p	$(c - C *_p)u(x, t)$	$u(-1, t) = u(1, t), \quad u_x(-1, t) = u_x(1, t)$
a	$(c - C *_a)u(x, t)$	$u(-1, t) = -u(1, t), \quad u_x(-1, t) = -u_x(1, t)$
N	$\sqrt{2}\{[(c - C *_p P_{\text{even}}) + (c - C *_a P_{\text{odd}})]u(x, t)\}$	$u_x(-1, t) = u_x(1, t) = 0$
D	$\sqrt{2}\{[(c - C *_p P_{\text{odd}}) + (c - C *_a P_{\text{even}})]u(x, t)\}$	$u(-1, t) = u(1, t) = 0$

Table 5.1: The choice of nonlocal operators based on the boundary conditions enforced.

Furthermore, the abstract convolutions, for fixed  $t \in J$ , are given in terms of the eigenbasis as follows:

$$\begin{aligned}
 C *_p u(x, t) &= \sum_{k \in \mathbb{Z}} \langle e_k^p | C \rangle_2 \langle e_k^p | u \rangle_2 e_k^p(x), \\
 C *_a u(x, t) &= \sum_{k \in \mathbb{Z}} \langle e_k^a | C \rangle_2 \langle e_k^a | u \rangle_2 e_k^a(x).
 \end{aligned}$$

For our numerical approximation of the solution of the nonlocal problem (5.1), we need to write these infinite sums as integral convolutions. This has been accomplished for all types of BCs in Section 4. We provide these integral convolutions below

$$\begin{aligned}
 C *_p u(x, t) &= \frac{1}{\sqrt{2}} \int_{-1}^1 \hat{C}_p(x-y) u(y, t) dy, \\
 C *_a u(x, t) &= \frac{1}{\sqrt{2}} \int_{-1}^1 \hat{C}_a(x-y) u(y, t) dy.
 \end{aligned}$$

## 5.1 Discretization in Space

To approximate the solution of (5.1) we begin with discretizing the domain  $\Omega$  into  $N$  subintervals by defining  $\Omega_h = \{K_1, K_2, \dots, K_N\}$  where  $K_i = (x_{i-1}, x_i)$  with  $-1 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1$ . We let  $h_i = |K_i| = x_i - x_{i-1}$  for  $i = 1, \dots, N$ . Given a polynomial degree  $\ell \geq 0$ , we wish to approximate the solution  $u(x, t)$  of (5.1) for a fixed  $t$  in the finite element space

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_\ell(K) \text{ for all } K \in \Omega_h\}$$

where  $P_\ell(K)$  is the space of polynomials of degree at most  $\ell$  on  $K$ .

We define the  $L^2$ -inner product on an element  $K \in \Omega_h$  as

$$(u, v)_K = \int_K u(x)v(x) dx \quad \text{and set} \quad (u, v)_{\Omega_h} = \sum_{K \in \Omega_h} (u, v)_K.$$

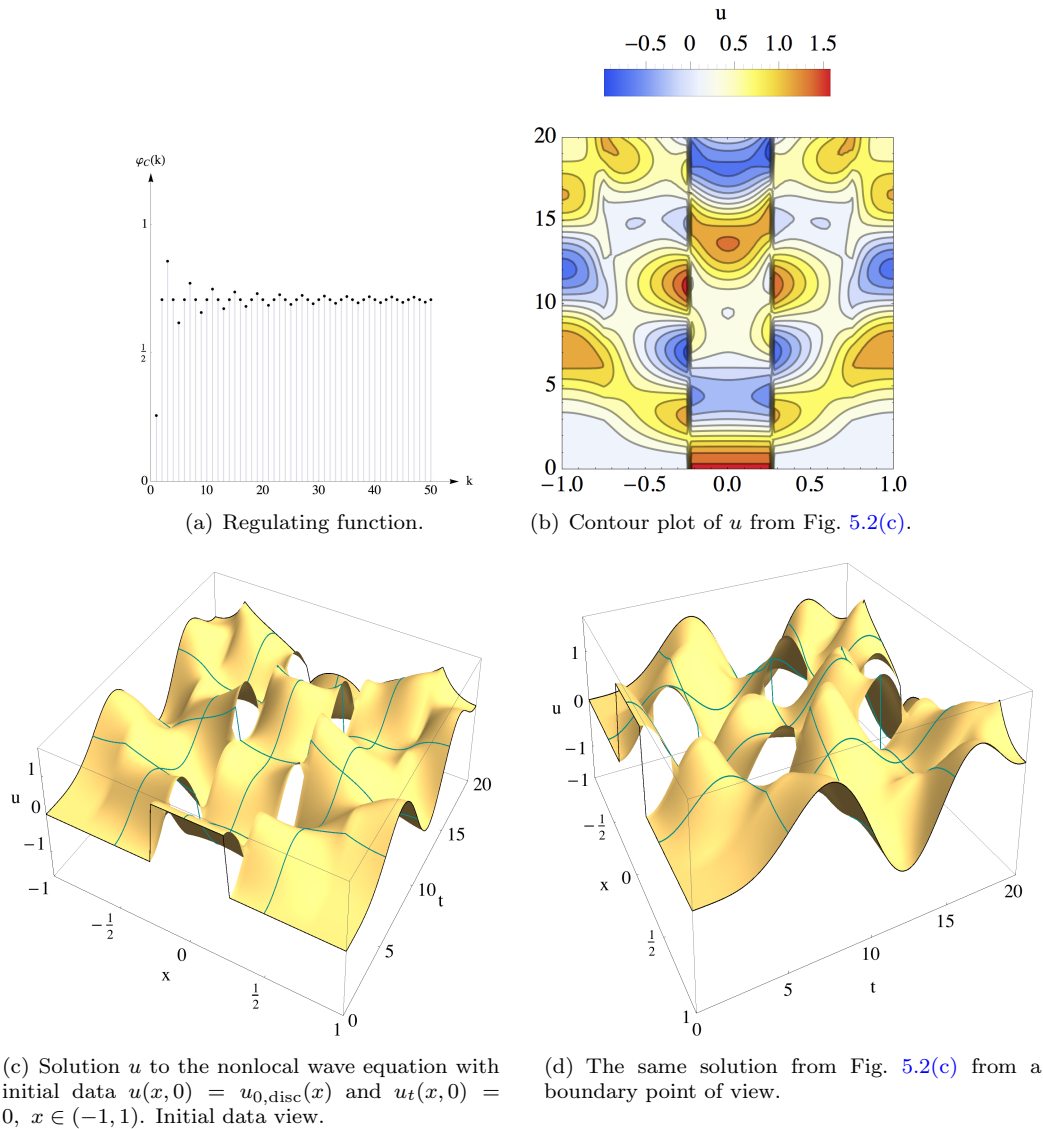


Figure 5.2: Solution to the nonlocal wave equation solution with periodic boundary conditions and vanishing initial velocity.

For approximation of (5.1), we use a Galerkin projection as used in [3, 4] and consider the following (semidiscrete) approximation: Find  $u^h : J \times V_h \rightarrow \mathbb{R}$  such that

$$(u_{tt}^h, v)_{\Omega_h} + (\varphi(A_{\text{BC}})u^h, v)_{\Omega_h} = (b, v)_{\Omega_h} \quad \text{for all } v \in V_h, \quad (5.3a)$$

$$u^h|_{t=0} = \Pi_h u_0, \quad (5.3b)$$

$$u_t^h|_{t=0} = \Pi_h v_0. \quad (5.3c)$$

Here,  $\Pi_h$  denotes the  $L^2$ -projection onto  $V_h$ .

## 5.2 Discretization in Time

The discretization of (5.1) by the Galerkin method (5.3) leads to the second-order system of ordinary differential equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}^h(t) = \mathbf{b}^h(t), \quad t \in J, \quad (5.4)$$

with initial conditions

$$\mathbf{M}\mathbf{u}^h(0) = \mathbf{u}_0^h, \quad \mathbf{M}\dot{\mathbf{u}}^h = \mathbf{v}_0^h. \quad (5.5)$$

Here,  $\mathbf{M}$  denotes the mass matrix and  $\mathbf{A}$  denotes the stiffness matrix. To discretize (5.4)-(5.5) in time, we employ the Newmark time-stepping scheme as described in [32], also see, e.g., [52].

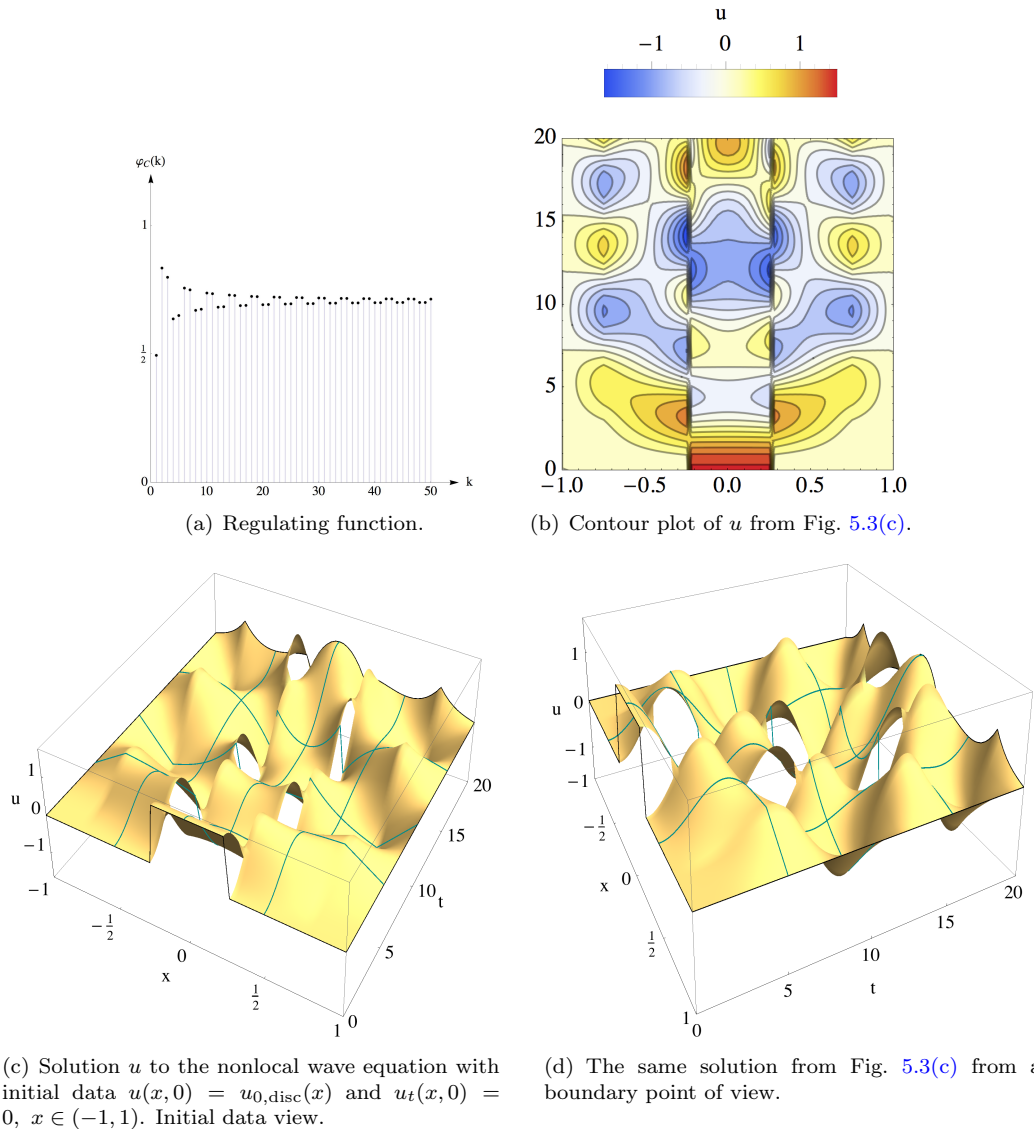


Figure 5.3: Solution to the nonlocal wave equation solution with antiperiodic boundary conditions and vanishing initial velocity.

Let  $k$  denote the time step and set  $t_n = n \cdot k$  for  $n = 1, 2, \dots$ . The Newmark scheme we employ consists in finding approximations  $\{\mathbf{u}_n^h\}_n$  to  $\mathbf{u}^h(t_n)$  such that

$$\mathbf{M}\ddot{\mathbf{u}}_1^h = \left(\mathbf{M} - \frac{1}{2}k^2\mathbf{A}\right)\mathbf{u}_0^h + k\mathbf{M}\mathbf{v}_0^h + \frac{1}{2}k^2\mathbf{b}_0^h, \quad (5.6)$$

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1}^h = (2\mathbf{M} - k^2\mathbf{A})\mathbf{u}_n^h - \mathbf{M}\mathbf{u}_{n-1}^h + k^2\mathbf{b}_n, \quad (5.7)$$

for  $n = 1, 2, \dots, N_t - 1$  where  $N_t k = T$ , and  $\mathbf{b}_n = \mathbf{b}(t_n)$ . Although there is a more general version of the Newmark time-stepping scheme, we made this particular choice due to the fact that it is second-order accurate and is explicit in the sense that at each time step we only have to solve a linear system with a coefficient matrix  $\mathbf{M}$  that is block diagonal. Hence,  $\mathbf{M}$  can be inverted at a very low computational cost. For other Newmark schemes the coefficient matrix of the linear system would be  $\mathbf{M} + k^2\beta\mathbf{A}$  for some  $\beta > 0$  which needs to be inverted at each time step. For a detailed discussion of more general Newmark time integration schemes we refer to [32].

### 5.3 Implementation Details

Let us describe a few details regarding the computation of the stiffness matrix  $\mathbf{A}$ . Let  $K \in \Omega_h$  and let  $\{\phi_j^K : j = 1, \dots, \ell + 1\}$  be a basis for  $P_\ell(K)$ . To fix ideas, let us consider the case  $\mathbf{BC} = \mathbf{p}$  so

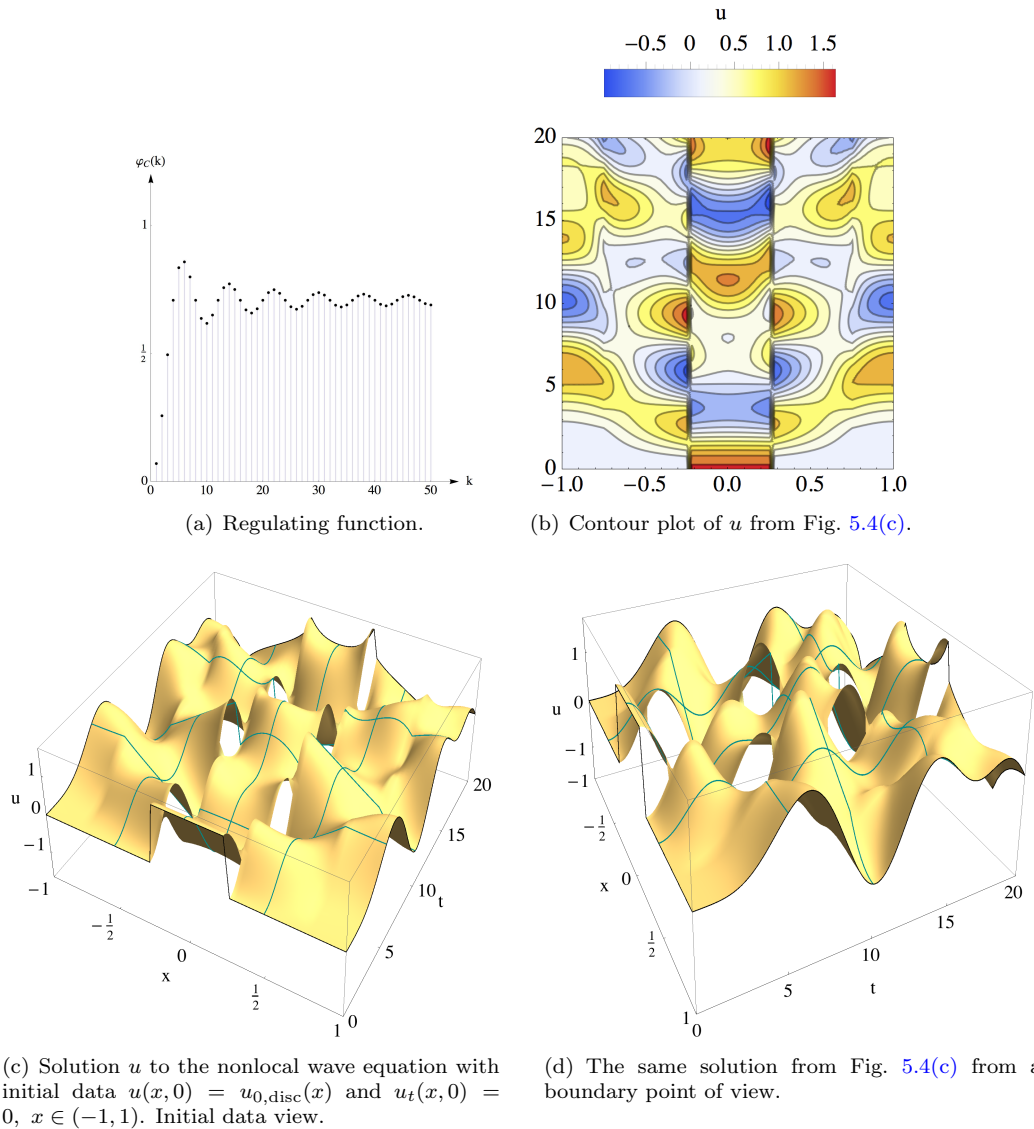


Figure 5.4: Solution to the nonlocal wave equation solution with Neumann boundary conditions and vanishing initial velocity.

that

$$\varphi(A_{\text{BC}})u(x, t) = (c - C *_p)u(x, t).$$

The remaining cases  $\text{BC} = \text{a}, \text{N}, \text{D}$  are similar.

First of all, we need to compute the constant  $c$  in (5.2). In the cases where  $C$  is an elementary function such as a (piecewise) polynomial, the exact value of this constant can be computed by direct integration. However, in the general case, we have to resort to numerical quadrature. We simply compute

$$c = \frac{1}{\sqrt{2}} \sum_{K \in \Omega_h} \int_K C(x) dx$$

where the integral on each element  $K \in \Omega_h$  is approximated by a quadrature rule. In this case, if  $C$  happens to have discontinuities or kinks in  $\Omega$ , in order to obtain an accurate approximation to  $c$ , we have to ensure that the nodes of the discrete domain  $\Omega_h$  are aligned with these discontinuities.

The matrix  $\mathbf{A}$  is of size  $N(\ell + 1) \times N(\ell + 1)$  and has a block structure. Each block-row of size  $(\ell + 1) \times N(\ell + 1)$  corresponding to an element  $K \in \Omega_h$  is determined by the equations

$$(\varphi(A_p)u^h, \phi_i^K)_K = (b, \phi_i^K)_K, \quad \text{for } i = 1, 2, \dots, \ell + 1.$$



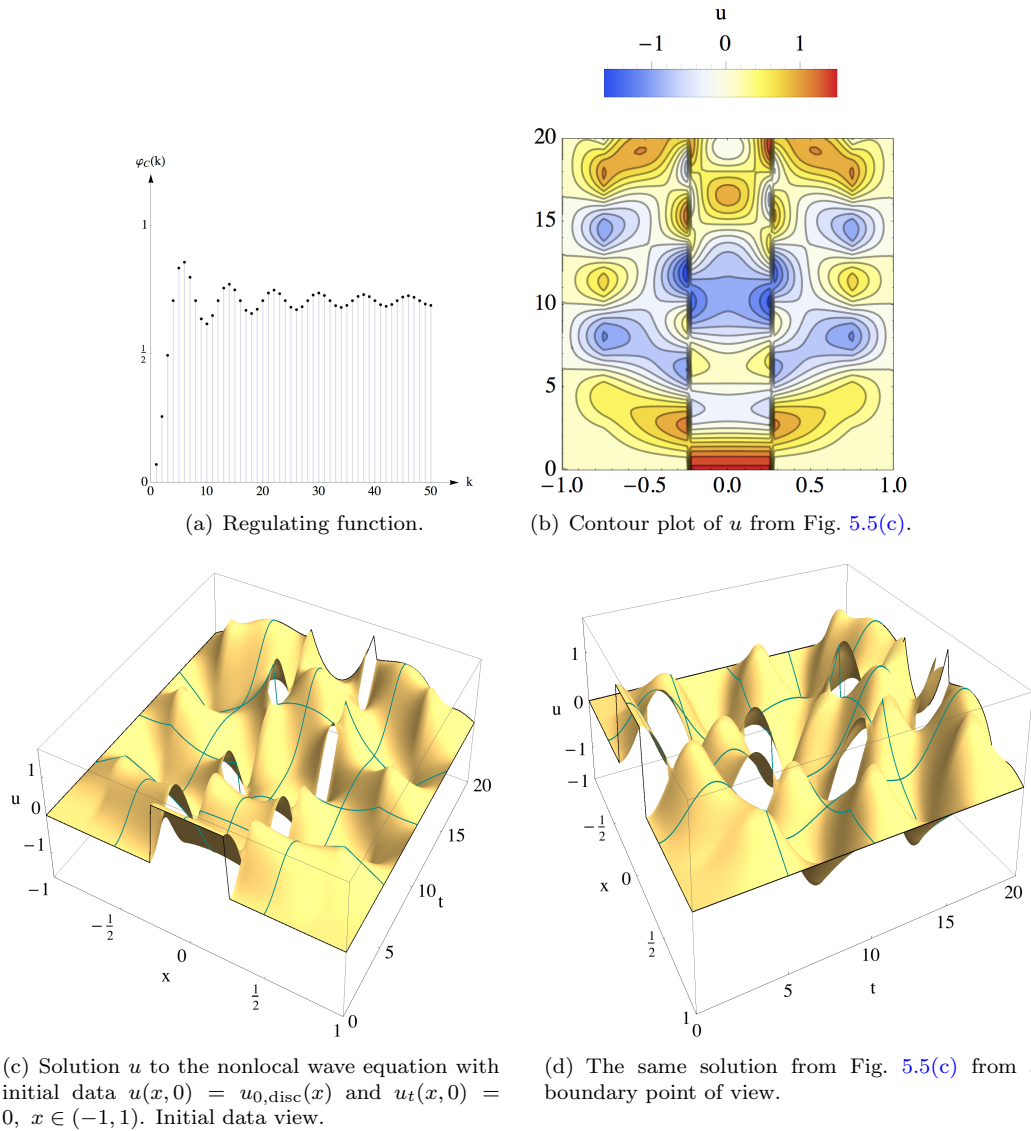


Figure 5.5: Solution to the nonlocal wave equation solution with Dirichlet boundary conditions and vanishing initial velocity.

Inserting the definition of  $\varphi(A_p)$ , we get

$$\begin{aligned} (\varphi(A_p)u^h, \phi_i^K)_K &= ((c - C *_p)u^h, \phi_i^K)_K \\ &= c(u^h, \phi_i^K)_K - (C *_p u^h, \phi_i^K)_K. \end{aligned}$$

The computation of the first term is standard, but we would like elaborate on a few details regarding the computation of the second term. At any fixed time  $t \in J$  and for a fixed element  $T \in \Omega_h$ , we have the restriction,  $u^{h,T}$ , of  $u^h$  on  $T$  has the expansion

$$u^{h,T}(x, t) = \sum_{j=1}^{\ell+1} u_j^T(t) \phi_j^T(x).$$

Then, since

$$\begin{aligned} C *_p u^h(x, t) &= \int_{-1}^1 \hat{C}_p(x - y) u^h(y, t) dy \\ &= \sum_{T \in \Omega_h} \sum_{j=1}^{\ell+1} u_j^T(t) \int_T \hat{C}_p(x - y) \phi_j^T(y) dy, \end{aligned}$$

we have

$$(C *_p u^h, \phi_i^K)_K = \sum_{T \in \Omega_h} \sum_{j=1}^{\ell+1} u_j^T(t) \int_K R_j^T(x) \phi_j^K(x) dx \quad (5.8)$$

where

$$R_j^T(x) := \int_T \hat{C}_p(x-y) \phi_j^T(y) dy.$$

Thus, we need to compute pointwise values of  $R_j^T$  which will be achieved through numerical quadrature. Note that the micromodulus function  $C$  may have points of discontinuities or kinks (or higher order derivatives of  $C$  may not be continuous) in  $\Omega$ . Hence, when computing  $R_j^T(x)$ , we need to take these points into account, for example, when using Gaussian quadrature which requires the smoothness of the integrand for optimal order accuracy. Furthermore, even if  $C$  is arbitrarily smooth in  $\Omega$ , its extension  $\hat{C}_p$  may not be smooth in  $[-2, 2]$ . Since the integrand involves  $\hat{C}_p(x-y)$  which is a translation of  $\hat{C}_p(-y) = \hat{C}_p(y)$  by  $x$  units to the left, we always have to account for possible singularities of  $\hat{C}_p(y)$  at the end points,  $\{-1, 1\}$ , of the domain  $\Omega$ . Suppose  $y_s \in T$  is such that  $\hat{C}_p(x-y_s)$  has a singularity in  $K$ . Then the integral defining  $R_j^T(x)$  has to be computed by writing  $T = T_1 \cup T_2$  where  $T = (x_L, x_R)$ ,  $T_1 = (x_L, y_s)$  and  $T_2 = (y_s, x_R)$ , and applying numerical quadrature on both subintervals. A similar treatment is needed when computing the integral  $\int_K R_j^T(x) \phi_j^K(x) dx$ .

Due to the nonlocal nature of the problem, the stiffness matrix  $\mathbf{A}$  is not necessarily sparse. This can be seen from (5.8) by observing that  $R_j^T$  does not necessarily vanish on the element  $K$  for  $T \neq K$ . The sparsity structure of  $\mathbf{A}$  is determined by the support of the micromodulus function  $C$ . More explicitly, the wider the support of  $C$ , the less sparse  $\mathbf{A}$  is. Symmetry and positive definiteness of the stiffness matrix are the consequences of the self-adjointness and positivity of the governing operator, respectively; see Theorem 12. For the case of periodic and Neumann BCs, the stiffness matrix becomes positive semidefinite and these systems can be solved by using numerical methods described in [13, 31]. Finally, we would like to point out that the assembly of the stiffness matrix as well as the mass matrix is independent of the time step and is performed only once.

#### 5.4 Approximations to Explicitly Known Exact Solutions

Note that, since the operator  $\varphi(A_{\text{BC}})$  is different for each BC,

In order to ascertain the convergence performance of the scheme described above, we display some numerical results corresponding to explicitly known exact solutions. We solve one example corresponding to each BC type. We take the exact solution corresponding to each BC as given in Table 5.2 and compute the corresponding right-hand side function  $b(x, t)$ .

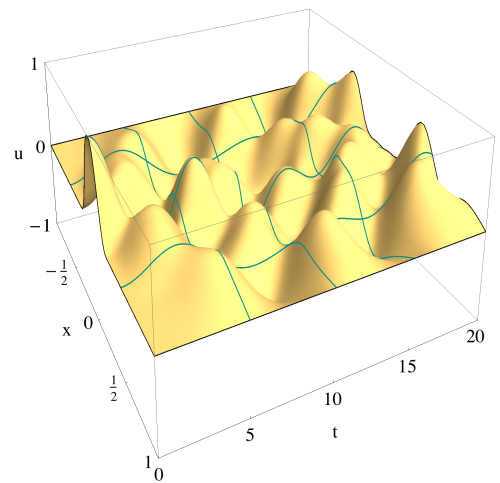
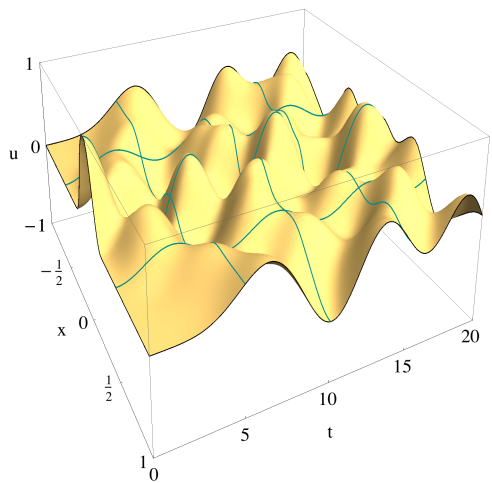
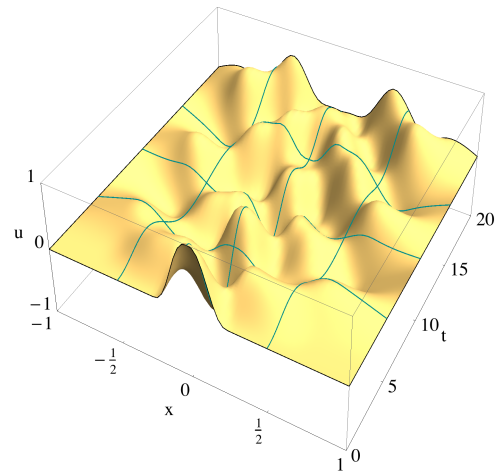
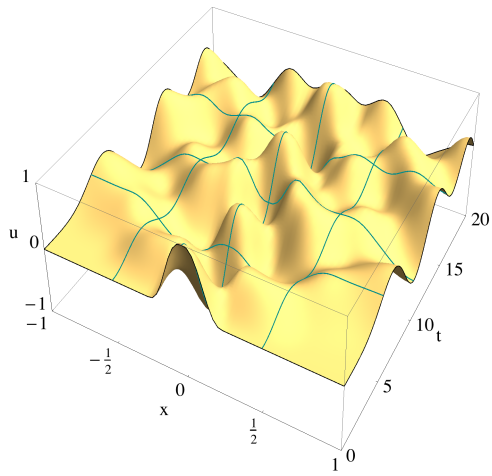
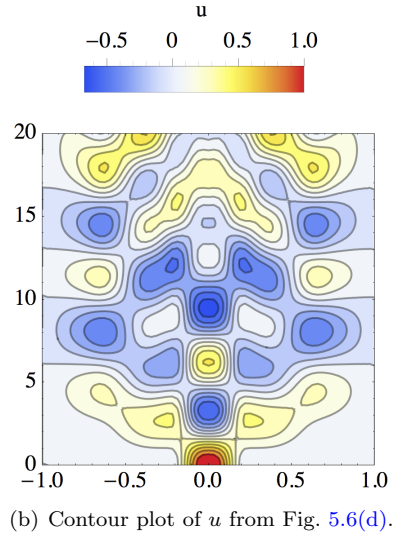
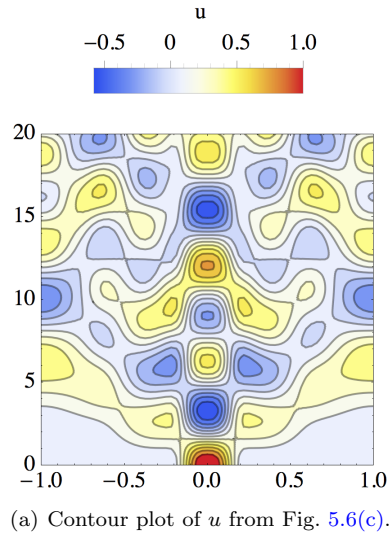


Figure 5.6: Solution to the nonlocal wave equation with Neumann ((a), (c), and (e)) and Dirichlet ((b), (d), (f)) boundary conditions, continuous initial displacement and vanishing initial velocity.

BC	$u(x, t)$
p	$t^2(\sin(\pi x) + \cos(\pi x))$
a	$t^2(x^4 - 1)$
N	$t^2((x^2 - 1)^2 - 8/15)$
D	$t^2(1 + \sin(\pi x) + \cos(\pi x))$

Table 5.2: Known exact solutions used in numerical experiments.

the corresponding source term  $b(x, t)$  is also expected to differ. We take the micromodulus function  $C$  to be the unit box on  $\Omega$ , namely,

$$C(x) = \begin{cases} 1, & x \in [-1/2, 1/2], \\ 0, & \text{otherwise,} \end{cases} \quad (5.9)$$

which is depicted in Figure 5.1.

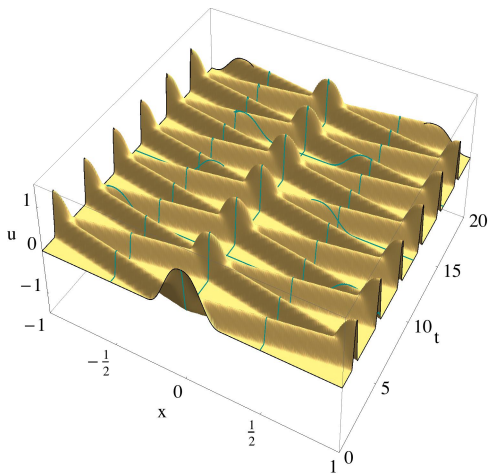
For each case, we compute the exact solution until the final time  $T = 20$  and compute the relative  $L^2$ -error  $\|(u - u^h)(T, \cdot)\|_0 / \|u(T, \cdot)\|_0$ . We first compute an approximate solution with a uniform coarse mesh with  $N = 2^3$  elements and then refine the mesh by subdividing each element into two elements of equal size. In each case, as the time step of the Newmark scheme we take  $\Delta t = 0.005$  so that the explicit Newmark time integration scheme is stable. In all of our examples, we found out that taking  $\Delta t$  so that  $\Delta t < h/10$  is sufficient for stability. Note that since the Newmark scheme is second order accurate in time, and all of the exact solutions in Table 5.2 is of the form  $u(x, t) = T(t)X(x)$  with  $T(t) = t^2$ , a second order polynomial, it is guaranteed that the dominant error is that in the space variable.

Table 5.3: History of convergence with known exact solutions for all BC types.

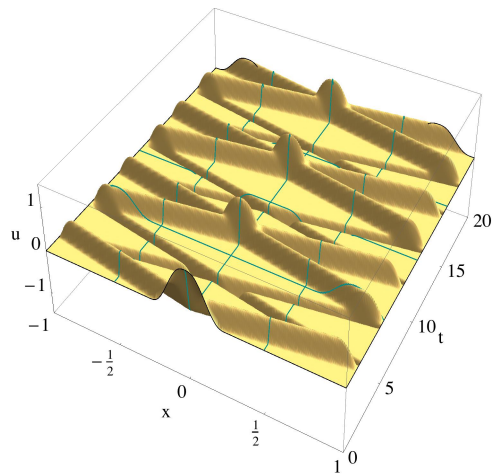
$\ell$	mesh	periodic		antiperiodic		Neumann		Dirichlet	
		error	order	error	order	error	order	error	order
0	3	2.32E-01	—	1.53E-01	—	2.34E-01	—	1.83E-01	—
	4	1.14E-01	1.02	6.88E-02	1.15	1.15E-01	1.03	8.35E-02	1.13
	5	5.68E-02	1.01	3.29E-02	1.06	5.72E-02	1.01	4.05E-02	1.04
	6	2.84E-02	1.00	1.62E-02	1.02	2.85E-02	1.00	2.01E-02	1.01
	7	1.42E-02	1.00	8.10E-03	1.00	1.43E-02	1.00	1.00E-02	1.00
1	3	2.28E-02	—	1.46E-02	—	2.30E-02	—	1.62E-02	—
	4	5.74E-03	1.99	3.69E-03	1.99	5.91E-03	1.96	4.06E-03	1.99
	5	1.44E-03	2.00	9.25E-04	2.00	1.49E-03	1.99	1.02E-03	2.00
	6	3.59E-04	2.00	2.32E-04	2.00	3.73E-04	2.00	2.54E-04	2.00
	7	8.98E-05	2.00	5.79E-05	2.00	9.32E-05	2.00	6.35E-05	2.00
2	3	1.52E-03	—	8.03E-04	—	2.05E-03	—	1.07E-03	—
	4	1.90E-04	2.99	1.01E-04	2.99	2.47E-04	3.05	1.35E-04	2.99
	5	2.38E-05	3.00	1.26E-05	3.00	3.06E-05	3.01	1.69E-05	3.00
	6	2.98E-06	3.00	1.58E-06	3.00	3.82E-06	3.00	2.11E-06	3.00
	7	3.73E-07	3.00	1.97E-07	3.00	4.77E-07	3.00	2.63E-07	3.00
3	3	7.51E-05	—	2.21E-05	—	5.03E-04	—	5.31E-05	—
	4	4.71E-06	3.99	1.38E-06	4.00	3.16E-05	3.99	3.33E-06	3.99
	5	2.95E-07	4.00	8.62E-08	4.00	1.98E-06	4.00	2.08E-07	4.00
	6	1.84E-08	4.00	5.39E-09	4.00	1.25E-07	3.99	1.30E-08	4.00

We display our numerical results in Table 5.3. Therein, the column labeled  $\ell$  indicates the polynomial degree we used to compute  $u^h$ , and the column labeled “mesh” denotes the mesh we used to compute the relative error displayed in the corresponding row, more explicitly, mesh=  $i$  means we used a uniform mesh with  $N = 2^i$  elements. In the column labeled “order” we display an approximate order of convergence as follows. If  $e_i$  denotes the relative error with mesh=  $i$ , then we display the quantity

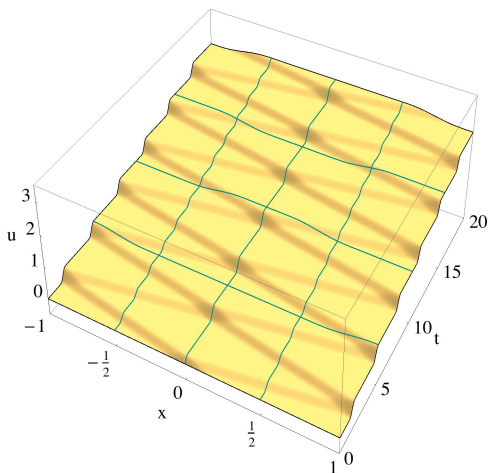
$$r_{i+1} = -\frac{1}{\log 2} \frac{e_{i+1}}{e_i}$$



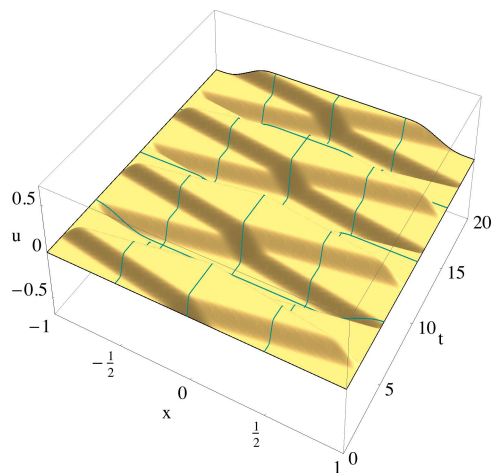
(a) Solution  $u$  to the classical (local) wave equation with initial data  $u(x, 0) = u_{0,\text{cont}}(x)$  defined in (5.11) and  $u_t(x, 0) = 0$ .



(b) Solution  $u$  to the classical (local) wave equation with initial data  $u(x, 0) = u_{0,\text{cont}}(x)$  defined in (5.11) and  $u_t(x, 0) = 0$ .



(c) Solution  $u$  to the classical (local) wave equation with initial data  $u(x, 0) = 0$  and  $u_t(x, 0) = u_{0,\text{cont}}(x)$  defined in (5.11).



(d) Solution  $u$  to the classical (local) wave equation with initial data  $u(x, 0) = 0$  and  $u_t(x, 0) = u_{0,\text{cont}}(x)$  defined in (5.11).

Figure 5.7: Solution to the classical wave equation with Neumann ((a) and (c)) and Dirichlet ((b) and (d)) boundary conditions with vanishing initial velocity ((a) and (b)) and vanishing initial displacement ((c) and (d)).

at the row corresponding to  $\text{mesh} = i + 1$ . The results displayed in Table 5.3 suggest an error estimate of the form

$$\frac{\|(u - u^h)(\cdot, T)\|_0}{\|u(\cdot, T)\|_0} \leq D h^{\ell+1}$$

for some constant  $D$  independent of  $u$  and  $h$ , that is, the method converges *optimally* with respect to the mesh size.

## 5.5 Approximations to Solutions

Here we display some numerical results in which we solve (5.3) with  $b = 0$  on  $\Omega \times J$ . In this case, we do not have an explicit representation of the solution and merely rely on numerical computing. We consider two initial displacement functions

$$u_{0,\text{disc}}(x) = \begin{cases} 3/2, & x \in [-1/4, 1/4], \\ 0, & \text{otherwise,} \end{cases} \quad (5.10)$$

and

$$u_{0,\text{cont}}(x) = \begin{cases} 0, & x \in (-1, -1/4), \\ (1+4x)^3(96x^2 - 12x + 1), & x \in [-1/4, 0), \\ (1-4x)^3(96x^2 + 12x + 1), & x \in [0, 1/4], \\ 0, & x \in (1/4, 1). \end{cases} \quad (5.11)$$

These functions are displayed in Fig. 5.1. In all cases, the initial velocity  $v_0(x) = 0$  for all  $x \in \Omega$ . The micromodulus function  $C(x)$  is again taken to be the unit box given in (5.9). We use the polynomial degree  $\ell = 2$  on a mesh with  $N = 64$  elements. For each BC case, we depict the regulating functions  $\varphi_C(k)$  utilized to define the nonlocal operator as well as the associated wave propagation; see Figures 5.2, 5.3, 5.4, and 5.5.

For  $t \in \mathbb{R}$ , we have proved that the solution is discontinuous if and only if the initial data is discontinuous; see Section 3.1. Furthermore, the position of discontinuity is determined by the initial data and should remain stationary. Since we use vanishing initial velocity, the explicit solution expression given in (3.1) is as follows:

$$u(x, t) = \left[ \cos \left( t \sqrt{\varphi(c)} \right) \right] u_0(x) + \delta u(x, t),$$

where  $\delta u(\cdot, t)$  is a continuous function for  $t \in \mathbb{R}$ . As seen in Figures 5.2, 5.3, 5.4, and 5.5, discontinuities of the initial data remain stationary at  $x = -1/4$  and  $x = 1/4$ . We also numerically verify that the prescribed BCs are satisfied for all  $t \in [0, 20]$ . For instance, it is easy to see that homogeneous Dirichlet BCs are satisfied in Figure 5.5. Furthermore, the governing operator preserves the reflection symmetry. In other words, since initial data (both  $u_{0,\text{disc}}$  and  $u_{0,\text{cont}}$ ) are symmetric with respect to  $x = 0$ , the displacement is symmetric with respect to  $x = 0$ , which can easily be observed by the symmetry in contour plots; see Figures 5.2(b), 5.3(b), 5.4(b), and 5.5(b).

We also report solutions of local and nonlocal equations with continuous initial data  $u_{0,\text{cont}}(x)$  given in (5.11). We observe several common features. Wave separation behavior is similar to that from the unbounded domain case as reported in the companion paper; see [12]. Namely, in the classical case, as expected, we observe the propagation of waves along characteristics; see Figure 5.7. In the nonlocal case, we observe oscillatory recurrent wave separation and oscillations are located at the center of the initial pulse. Hence, the wave patterns are symmetric with respect to  $x = 0$ ; see Figure 5.6. As far as the boundary behavior goes, in the classical case, we see that the Dirichlet BC creates reflections of opposite signs; see Figures 5.7(b) and 5.7(d). In the case of Neumann BC, reflections are of the same sign; see Figures 5.7(a) and 5.7(c). A parallel behavior is observed for the nonlocal Dirichlet case. Such parallel behavior is not obvious in the Neumann case. Further investigation of boundary behavior is needed.

## 6 Conclusion

This paper came about from the result in our companion paper [12] that the peridynamic governing operator is a bounded function of the classical governing operator on  $\mathbb{R}^n$ . The peridynamic operator contains a convolution. In this paper, we generalize convolutions to a bounded domain with the help of a eigenbasis obtained from classical operator on the bounded domain. This way, we can incorporate local boundary conditions into nonlocal governing operators. We study prominent BCs such as periodic, antiperiodic, Neumann, and Dirichlet. In the case of periodic and antiperiodic boundary conditions, integral representations of the abstract convolutions are relatively straightforward to establish. Such representations can also be achieved for the case of Neumann BCs, but with considerably more effort exploiting half-way symmetry. For Dirichlet BC, this integral representation involves an orthogonal projection of the micromodulus function onto a closed subspace defined in terms of the eigenbasis. We give an integral representation of this projection which does not depend on the eigenbasis. This representation involves a limit. Applying convolutions of the periodic and antiperiodic cases, we construct additional integral convolutions, what we call *simple convolutions*, respecting Neumann and Dirichlet BCs.

For the homogeneous nonlocal wave equation with the considered BCs, we prove that continuity is preserved by time evolution. Namely, for  $t \in \mathbb{R}$ , we prove that the solution is discontinuous if and only if the initial data is discontinuous. This is due to the fact that the solution has a unique decomposition into two parts. The first part is the product of a function of time with the initial data. This decomposition is induced by the fact that the governing operator has a unique



decomposition into multiple of the identity and a Hilbert-Schmidt operator; see (3.2). Hence, the second part is continuous. The decomposition also implies that discontinuities remain stationary. Whereas, in the classical case, it is well-known that discontinuities propagate along characteristics. We hold that this fundamentally difference is one of the most distinguishing features of PD.

The paper presents a unique way of combining the powers of abstract operator theory with numerical computing. The abstractness of the methods used in the paper allows generalization to other nonlocal theories. To substantiate the uniqueness of our treatment, we provide a comprehensive numerical study of the solutions of the nonlocal wave equation. We accomplish to demonstrate two crucial goals: For  $t \in \mathbb{R}$  and each BC considered, discontinuities of the initial data remain stationary and BCs are satisfied by the solution. We accomplish the two goals for all BCs and depict the corresponding solutions. For discretization, we employ a weak formulation based on a Galerkin projection and use piecewise polynomials on each element which allows discontinuities of the approximate solution at the element borders. We carry out a history of convergence study to ascertain the convergence behavior of the method with respect to the polynomial order and observe optimal convergence.

Generically, an operator with regular coefficients on  $\mathbb{R}^n$  has a purely discrete spectrum, providing an eigenbasis of the underlying space. The methods provided in this paper can treat problems in  $n$  spatial dimensions. To our knowledge, it is the first systematic approach of incorporating local BCs into nonlocal theories governed by bounded operators. Regulating functions are the key for capturing the essence of the underlying physics. The creation of a map from regulating functions to physical situations is an exciting research endeavor for future research. In conclusion, we believe that we added valuable tools to the arsenal of methods to treat nonlocal problems and compute their solutions.

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